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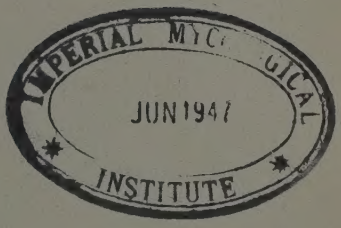

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MATHEMATICS

ON THE LOGARITHM OF AN INFINITELY DECOMPOSIBLE HERMITE-POSITIVE FUNCTION

By M. KREIN

(Communicated by A. N. Kolmogoroff, Member of the Academy, 8.III.1944)

As is well known, a complex-valued function $\Phi(p, q) \not\equiv 0$ of two arguments p and q running through a certain set of values Q is called an Hermite-positive nucleus, if for arbitrary $p_1 \in Q, \ldots, p_n \in Q$ $(n = 1, 2, \ldots)$ the form

$$\sum_{j,k=1}^{n} \Phi(p_j, p_k)\, \xi_j \overline{\xi}_k$$

is non-negative.

Denote by $\mathfrak{P}_a (0 \leqslant a \leqslant \infty)$ the aggregate of all continuous functions $f(x) (-a \leqslant x \leqslant a)$ * such that the function $f(x-y) (0 \leqslant x, y \leqslant a)$ is an Hermite-positive nucleus (and consequently, $f(-x) = \overline{f(x)}$ $(-a \leqslant x \leqslant a)$).

By $\mathfrak{G}_a (0 < a \leqslant \infty)$ denote the aggregate of continuous functions $g(x) (-a \leqslant x \leqslant a)$ such that $g(-x) = g(x) (-a \leqslant x \leqslant a)$ and the function

$$\Phi_g(x, y) = g(x-y) + g(0) - g(x) - g(-y) \quad (-a \leqslant x,\ y \leqslant a)$$

is an Hermite-positive nucleus.

It is evident that if $g(x) \in \mathfrak{G}_a$, then the function $g_1(x) = g(x) + \alpha + i\beta x$, where α, β are arbitrary real constants, also belongs to the class $\mathfrak{G}_a$, and that $\Phi_g \equiv \Phi_{g_1}$. It may be also easily shown that, conversely, if for any two functions $g \in \mathfrak{G}_a$, $g_1 \in \mathfrak{G}_a$ we have identically $\Phi_g = \Phi_{g_1}$, then $g_1(x) = g(x) + \alpha + i\beta x (-a \leqslant x \leqslant a)$, where α, β are real constants.

We shall call the function $f(x) \in \mathfrak{P}_a$ infinitely decomposible, if for any natural $n > 0$ there exists a function $f_n(x) \in \mathfrak{P}_a$ such that $f(x) \equiv [f_n(x)]^n$ $(-a \leqslant x \leqslant a)$. The aggregate of all infinitely decomposible functions $f(x) \in \mathfrak{P}_a$ we shall denote by $\mathfrak{P}^*_a$.

Observe that

$$\mathfrak{P}^*_a \subset \mathfrak{P}_a \subset \mathfrak{G}_a$$

The function $F(x) \in \mathfrak{P}_\infty$ is called a continuation into $\mathfrak{P}_\infty$ of the function $f(x) \in \mathfrak{P}_a$, if $F(x) \equiv f(x)$ $(-a \leqslant x \leqslant a)$. Similarly is established the notion of continuation of a function $f(x) \in \mathfrak{P}^*_a$ into $\mathfrak{P}^*_\infty$ and of continuation of a function $g(x) \in \mathfrak{G}_a$ into $\mathfrak{G}_\infty$.

It has been shown [1] that every function $f(x) \in \mathfrak{P}_a$ $(0 < a < \infty)$ has at least one continuation into $\mathfrak{P}_\infty$. Applying this fact and argueing on the lines similar to those which Schoenberg [2] used in dealing with a similar question, we prove without difficulties the following proposition.

---

* If $a = \infty$, then the values $x = \pm a$ must be, of course, excluded.

Theorem 1. *In order that a function* $f(x)$ $(-a \leqslant x \leqslant a)$ *should belong to the class* $\mathfrak{P}_a^*$, *it is necessary and sufficient that*

$$f(x) = \exp g(x) \quad (-a \leqslant x \leqslant a)$$

where $g(x)(-a \leqslant x \leqslant a)$ *is a certain function of the class* $\mathfrak{G}_a$.

Thus, if the function $f(x) \in \mathfrak{P}_a^*$, then $f(x) \neq 0$ everywhere in the interval $-a \leqslant x \leqslant a$.

Theorem 2. *In order that a function* $g(x)(-a \leqslant x \leqslant a)$ *should belong to the class* $\mathfrak{G}_a$, *it is necessary and sufficient that it should possess the representation*

$$g(x) = \alpha + i\beta x + \int_{-\infty}^{\infty} \left(e^{i\lambda x} - 1 - \frac{i\lambda x}{1+\lambda^2} \right) \frac{1+\lambda^2}{\lambda^2} d\sigma(\lambda) \quad (-a \leqslant x \leqslant a) \tag{1}$$

where $\sigma(\lambda)(-\infty < \lambda < \infty)$ *is a certain bounded non-decreasing function, and* α *and* β *are real constants.*

We shall norm the function $\sigma(\lambda)(-\infty < \lambda < \infty)$ by the conditions

$$\sigma(-\infty) = 0, \ \sigma(\lambda - 0) = \sigma(\lambda) \quad (-\infty < \lambda < \infty) \tag{2}$$

Theorem 2 is easily proved by means of Theorem 1 of our preceding note (3).

Combining Theorems 1 and 2 we obtain the representation of any function $f(x) \in \mathfrak{P}_a^*$ which was already established by Lévy (4) in the case $a = \infty$. The method of Khintchine (5) of proving P. Lévy's theorem enables us to obtain the required representation of the function $f(x) \in \mathfrak{P}_a^*$ also in the case $a < \infty$, which may be achieved in a comparatively simple way.

For real functions $g(x)$ the representation (1) in the assertion of Theorem 2 may be replaced by the following

$$g(x) = \alpha - \int_0^{\infty} \frac{\sin^2 \lambda x}{\lambda^2} (1+\lambda^2) d\tau(\lambda) \quad (-a \leqslant x \leqslant a)$$

where $\tau(\lambda)(0 \leqslant \lambda < \infty)$ is a bounded non-decreasing function, and α is a real constant. In this form Theorem 2 in the case $a = \infty$ must be considered as established by Neumann and Schoenberg (6).

From Theorem 2 immediately follows that every function $g(x) \in \mathfrak{G}_a$ has at least one continuation into $\mathfrak{G}_\infty$, and consequently also every function $f(x) \in \mathfrak{P}_a^*$ has at least one continuation into $\mathfrak{P}_\infty^*$.

Since for $a = \infty$ the non-decreasing function $\sigma(\lambda)$ [normed by the conditions (2)] is determined uniquely (4,5), every continuation $G(x) \in \mathfrak{G}_\infty$ of the function $g(x) \in \mathfrak{G}_a$ is generated by one and only one normal non-decreasing bounded function $\sigma(\lambda)$ yielding the representation (1) of the function $g(x)$. Denote by V_g the aggregate of all normed functions $\sigma(\lambda)$ yielding the representation (1) of a given function $g \in \mathfrak{G}_a$. Put for arbitrary real ξ

$$\rho(\xi) = \max [\sigma(\xi + 0) - \sigma(\xi - 0)] \quad (\sigma \in V_g)$$

It is evident that if V_g consists only of one function $\sigma(\lambda)$, then $\rho(\xi) = 0$ everywhere with the possible exception of a denumerable set of values ξ. The more interesting is

Theorem 3. *If the function* $g(x) \in \mathfrak{G}_a (a < \infty)$ *is continuable into* $\mathfrak{G}_\infty$ *not uniquely, then:*

1) $\rho(\xi) > 0 (-\infty < \xi < \infty)$ *and* $\rho^{-1}(\xi)$ *is an integral transcendental function of* ξ;

2) *for arbitrary real* ξ *there exists one and only one «canonic function»* $\sigma_\xi(\lambda) \in V_\varrho$ *such that*

$$\sigma_\xi(\xi+0)-\sigma_\xi(\xi-0)=\rho(\xi) \quad (-\infty<\xi<\infty).$$

3) *the points of increase of the canonic function* $\sigma_\xi(\lambda)$ *possess a unique point of condensation in infinity;*

4) *the points of increase of two different canonic functions* $\sigma_{\xi'}(\lambda)$ *and* $\sigma_{\xi''}(\lambda)(-\infty<\xi', \xi''<\infty)$ *are situated in such a way that each point of increase of one of them lies between two points of increase of the other.*

For want of space we omit the formulae by which the functions $\rho(\xi)$ and $\sigma_\xi(\lambda)$ may be computed and which enable us to obtain different characteristics of increase of the integral function $\rho^{-1}(\xi)$ and the characteristics of the density of the points of discontinuity of the functions $\sigma_\xi(\lambda)$.

Theorem 4. *In order that the function* $g(x) \in \mathfrak{G}_a$ $(a<\infty)$ *should be continuable into* $\mathfrak{G}_\infty$ *not uniquely, it is necessary and sufficient that for sufficiently small* $\varepsilon>0$ *the functions*

$$g(x)+\frac{1}{\varepsilon}, \quad g(x)+\varepsilon x^2 \quad (-a\leqslant x\leqslant a)$$

should belong, respectively, to the classes $\mathfrak{P}_a$ *and* $\mathfrak{G}_a$.

The arguments which we use in proving Theorem 4 enable us also to establish a new criterion of non-unique continuability of the function $f(x) \in \mathfrak{P}_a(a<\infty)$.

Theorem 5. *In order that a function* $f(x) \in \mathfrak{P}_a(a<\infty)$ *should be not uniquely continuable into* $\mathfrak{P}_\infty$, *it is necessary and sufficient that for sufficiently small* $\varepsilon>0$ *the functions*

$$f(x)-\varepsilon, \quad f(x)+\varepsilon x^2 \quad (-a\leqslant x\leqslant a),$$

should belong, respectively, to the classes $\mathfrak{P}_a$ *and* $\mathfrak{G}_a$.

In many cases it is convenient to use the following particular criterion.

Theorem 6. *If the function* $g(x)=\overline{g(-x)}$ $(-a\leqslant x\leqslant a,\ a<\infty)$ *possesses in the interval* $0\leqslant x\leqslant a$ *an absolutely continuous derivative*, then it belongs to the class* $\mathfrak{G}_a$ *in that and only that case, where for arbitrary continuous function* $\varphi(x)(0\leqslant x\leqslant a)$ *holds the inequality*

$$\int_0^a\int_0^a g''(x-y)\varphi(x)\overline{\varphi(y)}\,dx\,dy \leqslant -\Re\{g'(+0)\}\int_0^a |\varphi(x)|^2\,dx \tag{3}$$

If this condition is satisfied and, moreover, $\Re\{g'(+0)\}<0$**, *then in order that the function* $g(x)$ *should have a unique continuation into* $\mathfrak{G}_\infty$, *it is necessary and sufficient that the sign of equality in* (3) *should be attained for at least one continuous function* $\varphi(x) \not\equiv 0$.

We point out in passing that stated condition of unique continuability of the function $g(x) \in \mathfrak{G}_a$ is sufficient also in the case when $\Re\{g'(+0)\}=0$. It is also of interest that if the sign of equality in (3) is attained for $\varphi(x)=\varphi_0(x)$, $(\varphi_0(x) \not\equiv 0)$, then the points of increase of that unique function $\sigma(\lambda)$ which is involved in the representation (1) are the λ-roots of the equation

$$\int_0^a e^{i\lambda x}\varphi_0(x)\,dx=0.$$

---

* Under the derivative of $g(x)$ at the point 0 we understand the derivative from the right, $g'(+0)$.

** Observe that if (3) holds, we have always $\Re\{g'(+0)\} \leqslant 0$.

From Theorem 3 follows that if a certain function $g(x)=\overline{g(-x)}\,(-a\leqslant x\leqslant a)$ possesses in the interval $0\leqslant x\leqslant a$ an absolutely continuous derivative and $\Re\{g'(+0)\}<0$, then diminishing sufficiently the interval $(-a, a)$ we may assert that $g(x)\in\mathfrak{G}_a$ and is not uniquely continuable into $\mathfrak{G}_\infty$. Hence and from Theorem 1 follows that if a certain function $f(x)=\overline{f(-x)}\,(-a\leqslant x\leqslant a)$ possesses an absolutely continuous derivative in the interval $(0\leqslant x\leqslant a)$ and $f(0)>0$, $\Re\{f'(+0)\}<0$, then in the sufficiently diminished interval $-a_1\leqslant x\leqslant a_1$ the function $f(x)\in\mathfrak{P}^*_{a_1}$ and is not uniquely continuable into $\mathfrak{P}^*_\infty$.

In particular, for sufficiently small $a>0$ the function $f_a(x)=1-|x|\,(-a\leqslant x\leqslant a)$ belongs to the class $\mathfrak{P}^*_a$. On ground of Theorems 1 and 6 we may assert the existence of the maximal value of $a\,(<1)$ for which $f_a(x)\in\mathfrak{P}^*_a$. For this value of a the function $f_a(x)$ represents an example of a function which is uniquely continuable into $\mathfrak{P}^*_a$, but not uniquely continuable into $\mathfrak{P}_\infty$.

The following general fact may be proved: if a certain function $f(x)\in\mathfrak{P}^*_a$ is not uniquely continuable into $\mathfrak{P}^*_\infty$, then it always possesses continuations belonging to the class $\mathfrak{P}_\infty$, but not to the class $\mathfrak{P}^*_\infty$.

Observe that every continuous positive even function $f(x)\,(-a\leqslant x\leqslant a)$ belongs to the class $\mathfrak{P}_a$, if it is exponentially convex in the same interval. Since such a function may be always continued with preservation of the indicated properties, it is always not uniquely continuable into $\mathfrak{P}_\infty$, respectively into $\mathfrak{P}^*_\infty$.

It is known that the function $\varphi_\gamma(x;a)=\exp(-|x|^\gamma)\,(-a\leqslant x\leqslant a)$ belongs to $\mathfrak{P}^*_a$ for $0<\gamma\leqslant 2$ (for $\gamma>2$ this function does not belong even to the class $\mathfrak{P}_a$). It may be proved that for $0<\gamma<2$ this function is not uniquely continuable into $\mathfrak{P}^*_\infty$ (for $\gamma=2$ the function $\varphi_\gamma(x;a)$, being holomorphic at the point 0, is uniquely continuable even into $\mathfrak{P}_\infty$).

The theory of functions $g(x)$ of the class $\mathfrak{G}_a$ hangs closely together with the problem of continuation of helical arcs in Hilbert space [7]. All chief results of the present and the following note [7] are obtained by a simple application of author's investigation of the generalized problem of moments.

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MATHÉMATIQUES

QUELQUES THÉORÈMES DE LA MOYENNE DANS LES ESPACES À COURBURE CONSTANTE

Par M. OLEVSKY

(présenté par S. L. Soboleff, de l'Académie, le 12.III.1944)

Dans la présente Note en considérant la valeur moyenne d'une fonction sur une hypersphère dans l'espace non-euclidien R_n, nous donnons pour cet espace une généralisation de l'équation différentielle de Darboux (§ 1) et de la formule de Pizetti [1] (§ 2). A l'aide de ces formules plusieurs généralisations du théorème classique de Gauss sur la valeur moyenne des fonctions harmoniques [2-4] peuvent être étendues au cas des équations différentielles contenant le second paramètre différentiel de Beltrami (par rapport à l'élément linéaire de R_n) au lieu de l'opérateur de Laplace (§ 3—4). On obtient d'ailleurs des représentations intégrales et certaines relations pour les fonctions généralisées de Bessel [5] (§ 5).

§ 1. Soit

$$d\sigma^2 = \sum_{\alpha,\beta=1}^{n} g_{\alpha\beta}\, dx^\alpha\, dx^\beta \tag{1}$$

l'élément linéaire de l'espace R_n à $n \geqslant 2$ dimensions de courbure constante k.

La valeur moyenne $V(u], P, s)$ *de chaque fonction* u *deux fois dérivable sur une hypersphère* $F_s(P)$ *(au sens de la métrique* (1) *de* R_n*) de rayon* s *et de centre* P *vérifie l'équation*

$$\frac{\partial^2 V}{\partial s^2} + (n-1)\sqrt{k}\operatorname{ctg}\sqrt{k}\,s\,\frac{\partial V}{\partial s} = \Delta_2 V^* \tag{2}$$

et les conditions

$$V(u], P, s)|_{s=0} = u(P), \tag{3}$$

$$\left.\frac{\partial V}{\partial s}\right|_{s=0} = 0 \tag{4}$$

En rapportant $d\sigma^2$ aux coordonnées géodésiques-polaires, nous avons

$$d\sigma^2 = ds^2 + \left(\frac{\sin\sqrt{k}\,s}{\sqrt{k}}\right)^2 \sum_{i=1}^{n-1} \prod_{j=i+1}^{n-1} \sin^2\theta_j\, d\theta_i^{2**} \qquad \begin{matrix} 0 \leqslant s \leqslant \frac{\pi}{\sqrt{k}}; & k>0 \\ 0 \leqslant s < \infty; & k<0 \end{matrix}$$

$$0 \leqslant \theta_1 < 2\pi; \quad 0 \leqslant \theta_i \leqslant \pi \quad (i = 2, 3, \ldots, n-1)$$

---

* L'opérateur $\Delta_2 = \frac{1}{\sqrt{g}} \sum_\alpha \frac{\partial}{\partial x^\alpha}\left(\sum_\beta \sqrt{g}\, g^{\alpha\beta} \frac{\partial}{\partial x^\beta}\right)$ est pris par rapport aux coordonnées du point P.

** On pose $\prod_{i=m}^{n} \alpha_i$ égal à 1 si $m > n$.

L'élément de la surface $F_s(P)$ est $d\omega = \left(\frac{\sin\sqrt{k}s}{\sqrt{k}}\right)^{n-1} \sin\theta_2 \dots \sin^{n-2}\theta_{n-1}\, d\theta_1 \dots$ $\dots d\theta_{n-1}$; et l'élément de volume $dg = ds\, d\omega$. Formons $\partial V/\partial s$. Nous avons $\frac{\partial V}{\partial s} = \frac{1}{\omega} \int\limits_{F_s(P)} \frac{\partial u}{\partial n} d\omega$ (n est la direction de la normale). En appliquant le théorème de Gauss-Ostrogradsky étendu à l'espace R_n, on trouve

$$\frac{\partial V}{\partial s} = \frac{1}{\omega} \int\limits_{G_s(P)} \Delta_2 u\, dg \tag{a}$$

où $G_s(P)$ est le volume borné par $F_s(P)$. En dérivant encore une fois, on trouve

$$\frac{\partial^2 V}{\partial s^2} = \frac{\partial\,(1/\omega)}{\partial s} \int\limits_{G_s(P)} \Delta_2 u\, dg + \frac{1}{\omega} \int\limits_{F_s(P)} \Delta_2 u\, d\omega \tag{b}$$

$$\omega = \frac{2(\sqrt{\pi})^n}{\Gamma\left(\frac{n}{2}\right)} \left(\frac{\sin\sqrt{k}s}{\sqrt{k}}\right)^{n-1}$$

Il suit de (a) et (b)

$$\frac{\partial^2 V}{\partial s^2} + (n-1)\sqrt{k}\operatorname{ctg}\sqrt{k}s \frac{\partial V}{\partial s} = \frac{1}{\omega} \int\limits_{F_s(P)} \Delta_2 u\, d\omega \tag{2'}$$

Montrons que *l'opérateur Δ_2 est permutable avec l'opération qui consiste à prendre la moyenne sur $F_s(P)$*. Pour simplifier supposons $n=2$ et $k=-\alpha^2<0$. Prenons l'élément linéaire R_2 dans la forme $d\sigma^2 = \frac{dx_1^2 + dx_2^2}{\alpha^2 x_2^2}$. Alors la «circonférence» de «centre» $P(x_1; x_2)$ et de «rayon» z est une circonférence de centre $P'(x_1, x_2 \operatorname{ch}\alpha z)$ et de rayon $x_2 \operatorname{sh}\alpha z$. Donc

$$V(u], P, z) = \frac{x_2}{2\pi} \int_0^{2\pi} u(\xi, \eta) \frac{d\theta}{\eta}, \quad \Delta_2 V = \alpha^2 x_2^2 \left(\frac{\partial^2 V}{\partial x_1^2} + \frac{\partial^2 V}{\partial x_2^2}\right)$$

$$V(\Delta_2 u], P, z) = \frac{\alpha^2}{2\pi} \int_0^{2\pi} \eta^2 \left(\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2}\right) \frac{d\theta}{\operatorname{ch}\alpha z + \operatorname{sh}\alpha z \sin\theta}$$

où $x_1 + x_2 \operatorname{sh}\alpha z \cos\theta = \xi$, $x_2 \operatorname{ch}\alpha z + x_2 \operatorname{sh}\alpha z \sin\theta = \eta$. D'où il résulte, après quelques réductions, que l'on a

$$\delta \equiv \Delta_2 V(u]; P, z) - V(\Delta_2 u]; P, z) = \frac{\alpha^2}{2\pi} x_2 \int_0^{2\pi} \frac{d}{d\theta}\left(\frac{\partial u}{\partial \xi}\right) d\theta = 0$$

ce qu'il fallait démontrer. De ce que $\delta = 0$ et de (2') on déduit (2). Les conditions (3) et (4) résultent immédiatement de la définition $V = \frac{1}{\omega} \int\limits_{F_s(P)} u\, d\omega$ et de (a). En ce qui concerne la démonstration de l'unicité de la solution (2—4), il est commode de la considérer pour le problème isomorphe en prenant l'élément linéaire de R_n dans la forme riemannienne.

§ 2. En considérant (2—4) comme le problème de Cauchy pour $V(s)$, nous pouvons considérer Δ_2 comme un facteur symbolique constant par rapport à s. Quant à l'intégrale générale de l'équation $d^2V/ds^2 + (n-1)\sqrt{k}\operatorname{ctg}\sqrt{k}s(dV/ds) = \lambda V$, qui par la substitution $V\left(\frac{\sin\sqrt{k}s}{\sqrt{k}}\right)^{\frac{n-2}{2}} = S$ se réduit à l'équation considérée dans (5) [l'équation (1) pour $p = \frac{n-2}{2}$ et $\mu^2 = k\frac{n(n-2)}{4} - \lambda$], elle est de la forme $V = AJ_p(\mu; s) + BY_{p(\mu,s)}^{p(\mu,s)}$.

En utilisant (3) et (4), on trouve V d'abord dans la forme symbolique:

$$V(u],\ P,\ s) = \Gamma\left(\frac{n}{2}\right)\left[\left(\frac{\sin\sqrt{k}\,s}{2\sqrt{k}}\sqrt{k\frac{n(n-2)}{4} - \Delta_2}\right)^{-\frac{n-2}{2}} \times\right.$$

$$\left. \times J_{\frac{n-2}{2}}\left(\sqrt{k\frac{n(n-2)}{4} - \Delta_2},\ s\right)\right] u(P),$$

c'est-à-dire

$$V(u],\ P,\ s) = u(P) + \Gamma\left(\frac{n}{2}\right)\sum_{\nu=1}^{\infty}\left(\frac{\sin\frac{\sqrt{k}}{2}s}{\sqrt{k}}\right)^{2\nu} \times$$

$$\times \frac{\Delta_2(\Delta_2+nk)\ldots(\Delta_2+(\nu-1)(n+\nu-2)k)\,u(P)}{\nu!\,\Gamma\left(\frac{n}{2}+\nu\right)} \qquad (5)$$

Le développement (5) a lieu dans le domaine $G_s(P)$ appartenant à G, où u est une fonction analytique; dans le cas où u n'a qu'un nombre fini de dérivées dans G, on a au lieu de (5) une formule avec un reste. Pour $k \to 0$ on a la formule connue de Pizetti (1).

§ 3. Considérons l'équation

$$\sum_{i=0}^{m} c_i \Delta_2^i u = 0 \qquad (6)$$

(où les c_i sont des constantes). Il suit de la formule (5) que la valeur moyenne $V(u], P, s)$ de la solution u de l'équation (6) est linéaire par rapport à $u(P)$, $\Delta_2 u(P)$, ...; $\Delta_2^{m-1}u(P)$. La formule finale peut être représentée de la manière suivante*:

$$\begin{vmatrix} V(u],\ P;\ s) & S(s,\lambda_1) & \ldots & S(s;\ \lambda_m) \\ u(P) & 1 & \ldots & 1 \\ \Delta_2 u(P) & \lambda_1 & \ldots & \lambda_m \\ \ldots & \ldots & \ldots & \ldots \\ \prod_{i=0}^{m-2}(\Delta_2 + a_i k)\,u(P) & \prod_{i=0}^{m-2}(\lambda_1 + a_i k)\ldots & & \prod_{i=0}^{m-2}(\lambda_m + a_i k) \end{vmatrix} = 0 \qquad (7)$$

où $\lambda_1, \ldots, \lambda_m$ sont les racines du polynôme $\sum_{i=0}^{m} c_i z^i$; $a_i \equiv i(n+i-1)$, et

$$S(s,\lambda_i) = \Gamma\left(\frac{n}{2}\right)\left(\mu_i\frac{\sin\sqrt{k}\,s}{2\sqrt{k}}\right)^{-p} J_p(\mu_i, s)$$

$$\mu_i = \sqrt{kp(p+1) - \lambda_i}, \quad p = \frac{n-2}{2}$$

Pour $\lambda_i = \lambda_{i-1}$ on a dans la formule (7) au lieu de la colonne $S(s,\lambda_{i-1})$, $1, \lambda_{i-1}, \ldots$ la colonne $\frac{\partial S(s,\lambda_i)}{\partial \lambda_i}$, $0, 1, \ldots$

En particulier, si $\lambda_1 = \lambda_2 = \ldots = \lambda_m$, c'est-à-dire (6) prend la forme

$$(\Delta_2 - \lambda)^m u = 0 \qquad (8)$$

la formule (7) se transforme en

$$V(u],\ P,\ s) = S(s,\lambda)\,u(P) + \sum_{i=1}^{m-1}\frac{1}{i!}\frac{\partial^i S(s,\lambda)}{\partial\lambda^i}(\Delta_2 - \lambda)^i u(P) \qquad (9)$$

* La formule (7) peut être écrite dans une forme qui ne contient pas d'opérations différentielles par rapport à u, si l'on utilise les moyennes d'ordre supérieur à 1, analogues à celles qui ont été introduites (dans E_n) par Nicolesco(2).

Pour $m=1$ on a la généralisation immédiate de la formule classique (pour $k \to 0$) de Weber (3).

§ 4. A l'aide de l'équation de Darboux (2) et du théorème d'unicité correspondant à (2—4) on peut établir le théorème suivant:

La valeur moyenne de la fonction $u(x_1, x_2, \ldots, x_n; y_1, y_2, \ldots, y_n) \equiv u(x, y)$, *vérifiant l'équation*

$$\underset{x}{\Delta_2} u = \underset{y}{\Delta_2} u \tag{10}$$

(*où* $\underset{z}{\Delta_2}$ *est pris par rapport à l'élément* $d\sigma^2 = \sum_{\alpha,\beta=1}^{n} g_{\alpha\beta}(z)\, dz^\alpha dz^\beta$ *de l'espace* R_n *à courbure constante* k) *sur une «sphère» de «centre»* $P(x_1, x_2, \ldots, x_n)$ *et de «rayon»* z, *est égale à la valeur moyenne de cette fonction sur la «sphère» de «centre»* $Q(y_1, y_2, \ldots, y_n)$ *et de même «rayon»* z

$$\int_{F_z(P)} u(\xi, y)\, d\omega_\xi = \int_{F_z(Q)} u(x, \eta)\, d\omega_\eta \tag{11}$$

Pour $k=0$ nous obtenons le théorème d'Asgeirsson *.

§ 5. Les théorèmes du § 3 jouissent de cette propriété que non seulement leurs inversions donnent des caractéristiques intégrales pour les solutions de l'équation (6), mais ils conduisent immédiatement à une serie de représentations intégrales pour les fonctions généralisées de Bessel et donnent des relations intégrales entre ces fonctions.

Utilisons, par exemple, en ce sens la formule de Weber que nous avons généralisée (9). Soit $k<0$; écrivons l'équation (8) pour $m=1$ en prenant $d\sigma^2$ dans la forme $d\sigma^2 = \frac{dx_1^2 + \ldots + dx_n^2}{x^2 x_n^2}$; $k=-\alpha^2$ (voir § 1). Sa solution, qui ne dépend que de x_n, a la forme: $u = C_1 x_n^{r_1} + C_2 x_n^{r_2}$, où r_1 et r_2 sont les racines de l'équation $r^2-(n-1)r-\frac{\lambda}{\alpha^2}=0$. En appliquant (9) à $u=x_n^{r_1}$ (il faut tenir compte de ce que l'élément de la surface sphère est pris dans la métrique non-euclidienne) nous réduisons facilement l'intégrale de surface à l'intégrale simple et nous obtenons:

$$\int_0^\pi (\operatorname{ch}\alpha z + \operatorname{sh}\alpha z \cos\theta)^{-\frac{n-1}{2}+\sqrt{\frac{1}{4}-\frac{\mu^2}{\alpha^2}}} \sin^{n-2}\theta\, d\theta = \sqrt{\pi}\,\Gamma\left(\frac{n-1}{2}\right)\left(\frac{\mu \operatorname{sh}\alpha z}{2\alpha}\right)^{-\frac{n-2}{2}} J_{\frac{n-2}{2}}(\mu, z) \tag{12}$$

Pour $n=2$ l'égalité (12), si l'on y pose $\mu^2 = -\alpha^2\nu(\nu+1)$, prend la forme de la représentation intégrale bien connue (6) de Laplace pour la fonction de Legendre $P_\nu(\operatorname{ch}\alpha z)$.

Les autres représentations intégrales et relations peuvent être obtenues dans la même voie, si l'on prend Δ_2 dans (8) en d'autres systèmes de coordonnées.

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* Asgeirsson (4) a tiré de son théorème une méthode très simple pour résoudre le problème de Cauchy pour l'équation des ondes $\Delta u = u_{tt}$. Il est essentiel de remarquer que ceci n'est possible que pour l'espace euclidien E_n, car il est facile de démontrer que l'équation des ondes dans R_n ($\Delta_2 u = u_{tt}$) peut être considérée comme un cas particulier de l'équation (10) seulement pour $k=0$.

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MECHANICS

ON EVOLUTES OF CONNECTING-ROD CURVES

By I. I. ARTOBOLEVSKY, Corresponding Member of the Academy

The study of the curvature of connecting-rod curves is of great importance for solving problems presented by the synthesis of mechanical apparatus, particularly so in the case of intermittent motion mechanisms. Owing to the complexity of connecting-rod curve equations, the analytical method is very seldom employed in studying the curvature of these curves; much the most

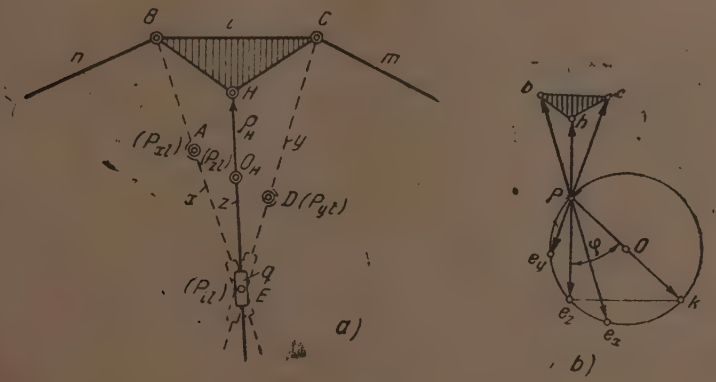

Fig. 1

often the problem is solved graphically by studying the acceleration of some points of the connecting-rod plane and the relation of their respective accelerations to the radii of curvature of their trajectories.

In the present paper a new geometric method is proposed, whereby problems on the curvature of connecting-rod curves are dealt with by means of velocity diagrams and the loci described in an earlier work (1).

Suppose we have to determine the centre of curvature O_H and radius of curvature ρ_H of a point H of an arbitrary link i which as a part of a mechanism makes up kinematic pairs with links n and m (Fig. 1,*a*). As has been shown earlier (1), the motion of the link i can be represented as the motion of a connecting-rod BC in a four-bar chain $ABCD$, in which the points A and D are instantaneous centres of rotation P_{xl} and P_{yl} of added links x and y with respect to an immovable link l. Let an additional link z be attached at point H to the connecting-rod BC so that its one end forms a pair H with the connecting-rod, and the other end will always pass through the point E to a hinge attached in addition to links x and y of the group (2). This is easily achieved by mounting a slide block q with a hinge at point E (Fig. 1,*a*).

What we are now to prove is that when the whole mechanism is set in motion, the centre of curvature O_H will always coincide with the instantaneous centre of rotation P_{zl} of the link z. Let us plot a diagram of velocities pbc of the mechanism $ABCD$ (Fig. 1, b) and draw in it the above-mentioned circumference with centre at O.

According to Savari's theorem (3), the angular velocity $\bar{\omega}_i$ of the link i, the velocity $\bar{u}$ of the change of the instantaneous centres of rotation, the distance $P_{il}H$, the angle φ (Fig. 1, b) and the radius ρ_H are all involved in the following relation:

$$\frac{\omega_i}{u} = \cos\varphi \left(\frac{1}{P_{il}H - \rho_H} - \frac{1}{P_{il}H} \right) \tag{1}$$

Hence

$$\rho_H = \frac{\omega_i (P_{il}H)^2}{u\cos\varphi + \omega_i (P_{il}H)} \tag{2}$$

As we have $\omega_i (P_{il}H) = v_H = \mu_v (ph)$ where $\bar{v}_H$ is the velocity of the point H and μ_v is the scale of the velocity diagram, and also have $u\cos\varphi = \mu_v (pk)\cos\varphi = \mu_v (pe_z)$ (Fig. 1, b), therefore

$$\rho_H = \frac{(ph)}{(pe_z)+(ph)} (P_{il}H) = \frac{(ph)}{(e_z h)} (P_{il}H) = \frac{(ph)}{(pe_z)} O_H E \tag{3}$$

It follows from (3) that in order to find the centre of curvature O_H and the radius of curvature ρ_H one has to divide the intercept HE of the link z

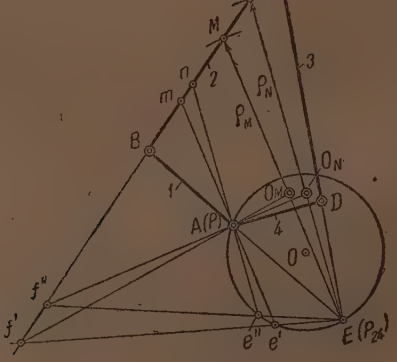

Fig. 2

into portions proportional to the intercepts (ph) and (pe_z) in the velocity diagram.

As the intercepts (ph) and (pe_z) are proportional to the velocities of the points H and E_z of the link z, the division just mentioned reduces to finding on the link z a point coinciding with the instantaneous centre of rotation P_{zl} of the link z. Thus, at any position of the mechanism, the centre of curvature O_H of the trajectory of the point H will coincide with the position of the instantaneous centre of rotation of the added link z. In other words, the problem of finding the centre of curvature is reduced to that of finding the instantaneous centre of rotation, and this latter can always be solved by means of an ordinary velocity diagram.

The geometric method of finding the curvature centres for different points of a connecting-rod BC of four-bar chain $ABCD$ is shown in Fig. 2. To find the centre of curvature O_M of the trajectory of the point M we draw a circumference about O and connect the point M with point $E(P_{34})$. We draw a straight line me' through point $A(p)$, parallel to ME, until it crosses the centre O circumference at point e'. A straight line drawn from E to e' is produced until it meets at f' with the produced axis BC of the connecting-rod. The point f' is connected with point $A(p)$ by a line which meets the line ME at point O_M. By construction

$$\frac{O_M M}{O_M E} = \frac{pm}{pe'}$$

in other words, point O_M coincides with the instantaneous centre of rotation of the link ME and is, therefore, the centre of curvature of the trajectory of point M when the mechanism is in the position considered here. The intercept $O_M M$ is the radius of curvature ρ_M. For finding the centre of curvature O_N of the trajectory of some other point, say N, we have to find in a similar way the points n, e'' and f''. Drawing a straight line from f'' to $A(p)$ we get at its intersection with the line NE the centre of curvature O_N; and so on.

Fig. 3

The property we have proved permits the following generalization to be made. Any point of a mechanism link describing a connecting-rod curve has the evolute of its trajectory representable as a stationary centroid of an added link to the mechanism. In Fig. 3 we have a four-bar chain $ABCD$ whose point M describes a connecting-rod curve $\alpha-\alpha$. The locus of instantaneous centres of rotation of an added link ME forms a stationary centroid C_H-C_H which is just the evolute of the curve $\alpha-\alpha$. The moving centroid $C_\Pi-C_\Pi$ is the link ME itself. Thus, as the point M moves along its trajectory $\alpha-\alpha$, the link ME rolls without sliding along the curve C_H-C_H.

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HYDROMECHANICS

ON THE SOLUTION OF PROBLEMS OF GROUND-WATER MOTION BY THE METHOD OF P. J. POLUBARINOVA-KOTSCHINA

By N. K. KALININ

(Communicated by V. I. Posdunine, Member of the Academy, 23.X.1943)

1. Let the movement of ground-water be steady, plane (parallel to the vertical plane) and subject to Darcy's law. The ground is homogeneous and isotropic. Neither the fluid nor the ground are compressible. Suppose, moreover, that the region of the flow is simply connected* (and one-sheeted) and its boundary composed of rectilinear segments (solid walls, water reservoir boundaries) and a free surface, at all of which certain boundary conditions are fulfilled ([1,2]).

On these assumptions the problem of ground-water movement is known to consist in finding a complex potential $f=\varphi+i\psi$ regular ([1]) within the region of the flow (we refer this region to the plane of the complex variable $z=x+iy$) and satisfying at its borders some boundary conditions. In case a free surface is present, its position has to be determined, too.

Mapping conformably the flow region on to a region of definite shape (circle or upper half-plane) belonging to a new auxiliary variable t, we can put the problem of searching in this region for the functions

$$z(t),\ f(t) \tag{1}$$

or

$$Z=dz/dt,\quad F=df/dt \tag{2}$$

regular in it and satisfying on its border certain boundary conditions. The solution of the problem in the form (1) and (2) will be described as parametric.

The method of Polubarinova-Kotschina ([3-5]) is to obtain a parametric solution in the form (2) when the upper half-plane t is taken for the auxiliary region T and the functions (2) are sought for as integrals of a homogeneous linear differential equation

$$\frac{d^2U}{dt^2}+P(t)\frac{dU}{dt}+Q(t)U=0 \tag{3}$$

Thus, the problem breaks up in two parts. First, the differential equation (3) must be constructed. This can, as has been shown by Polubarinova-Kotschina and further proved by Riesenkampf, be done when the singularities of functions (2) are known.

* For the time being the region is supposed for simplicity to contain no point sources (drains).

Secondly, the constructed equation is to be solved and the boundary conditions satisfied, which, under the assumptions made, reduce on separate segments of the real axis t to two conditions of the following kind: the imaginary part of some linear combination of functions (2) with constant coefficients is equal to zero [3].

In the filtration theory the construction of the equation (3) is facilitated by the fact that it belongs [3-5] in the Fuck's class of equations.

To build up the equation (3) correctly, all its singular points and the indices of these points must be known. If, owing to the works of Polubarinova-Kotschina and Riesenkampf, who have suggested a very simple geometric method, the determination of the indices is essentially a settled problem, nevertheless, the problem of finding all singular points of equation (3) is not altogether clear.

Below it will be shown that all singular ponts of equation (3) can be found if we know the travel-time curve (on the assumptions made at the beginning of this article the travel-time curve can generally be constructed and its outline will be found to consist of straight segments and arcs). Once the indices of these points have been determined, it is possible, following certain rules [5], to construct the equation (3) corresponding to a given filtration problem.

In proving this statement we will rest upon the following three theorems established by Riesenkampf for the singular «by-points» (Nebenpunkte, nebenpoints) of equation (3).

As nebenpoints are described those singular points of equation (3), at which all its solutions are regular.

Theorem 1. *In the neighbourhood of an unremovable nebenpoint* $t=t_0$ *equation* (3) *has a canonic system of solutions of the form*

$$U_1=(t-t_0)^n[a_0+a_1(t-t_0)+\ldots],\ (a_0\neq 0) \tag{4}$$

$$U_2=(t-t_0)^{n+\nu}[b_0+b_1(t-t_0)+\ldots]\ (b_0\neq 0) \tag{5}$$

where n *and* ν *are integers,* $n\geqslant 0$, $\nu\geqslant 2$. *If* $\nu=1$, $n=0$, *the point* $t=t_0$ *is an ordinary point. For* $\nu=1$, $n>0$ *this point can be removed from equation* (3) *by proceeding to a new function* V *according to the formula*

$$U_i=(t-t_0)^n V$$

For a differential equation satisfied by V *the point* $t=t_0$ (*for* $\nu=1$) *is an ordinary point.*

Theorem 2. *To the neighbourhood of the nebenpoint* t_0 *of the plane* t *there corresponds on the plane* $\frac{df}{dz}$ *and, accordingly, on the travel-time curve, a* ν*-sheeted mapping of the flow with a* $\nu-1$ *multiple point of branching. And conversely, to any branching point of the travel-time curve for which the solutions* U_1 *and* U_2 *are regular, there corresponds a nebenpoint of equation* (3).

Theorem 3. *Nebenpoints not lying on the real axis of the* t *plane are arranged symmetrically with respect to this axis.*

Now it is easy to show the truth of the following

Lemma 1. *Every point* t_0 *of the upper half-plane* t ($\mathrm{Im}\, t_0>0$) *that corresponds to no branching point of the travel-time curve is an ordinary point of the differential equation* (3).

As, in fact, the functions $Z=dz/dt$, $F=df/dt$ which by condition are linearly independent integrals of equation (3), are regular for $t=t_0$, this point cannot but be a nebenpoint of equation (3), that is, in its neighbourhood the canonic system of solutions will have the form (4), (5).

But as the point $t = t_0$ fails by assumption to correspond to a branching point on the travel-time curve and its neighbourhood is accordingly a one-sheeted mapping of the neighbourhood of the respective point of the travel-time curve, therefore $\nu = 1$ by virtue of Theorem 2. On the other hand, the canonic form of one of the integrals is easily found; if we note that; the flow being assumed to be one-sheeted, we have for $t = t_0$ the expression

$$U_1 = dz/dt = a_0 + a_1(t - t_0) + \ldots \quad (a_0 \neq 0) \qquad (6)$$

Comparing it with (4) we find that $n = 0$. Hence, by Theorem 1, the point $t = t_0$ is an ordinary point in equation (3), q. e. d.

Marking out in the upper half-plane of t those points which correspond to the branching points of the travel-time curve, and taking their symmetrical points in the lower half-plane, we shall obtain; in accordance with Theorem 3 and the Lemma 1; all singular points of equation (3) lying outside the real axis of the variable.

If; in particular, the travel-time curve is a region without branching points; then all singular points of equation (3) *will lie on the real axis t.*

Let us now proceed to the singular points of equation (3) lying on the real axis t.

As s i n g u l a r we describe the following points of the travel-time curve: 1) angular points (among their number those removed to infinity); 2) non-angular; which are either junction points of parts along which various boundary conditions are fulfilled for the functions (1) or correspond to infinitely remote points of the flow contour (in the z plane).

We may establish

L e m m a 2. *The points of the real axis t corresponding to no singular points of the travel-time curve are ordinary points in equation* (3).

Let us consider; indeed; some point t_0 of this type and denote by A_0 the corresponding (ordinary) point of the travel-time curve . It is known [3] that the functions (2) can be continued analytically into the lower half-plane through that segment of the real axis t which contains the point t_0 and will therefore be regular in the neighbourhood of this point. Thus; the point t_0 is bound to be a nebenpoint of equation (3). By virtue of the assumptions made, the function dz/dt can in the neighbourhood of $t = t_0$ be developed in the form (6). On the other hand; the above continuation of the functions (2) will be parallelled on the travel-time curve by an extension of the domain of values of df/dz through the segment of the travel-time curve containing the point A_0. Since by virtue of the assumptions made in this paper the travel-time curve consists, as has been mentioned above, of rectilinear segments and circular arcs, the extension just mentioned can be carried out on the symmetry principle of Riemann-Schwarz. The point A_0, owing to its character, cannot possibly be a branching point of the extended domain of the travel-time curve.

Hence; according to Lemma 1, the point $t = t_0$ is an ordinary point relative to equation (3).

From the lemmas we have established there follows

T h e o r e m. *No other points but the following ones are singular points of equation* (3): 1) *the points of the upper half-plane t, corresponding to the branching points of the travel-time curve*; 2) *their symmetrical points with respect to the real axis in the lower half-plane*; 3) *the points of the real axis t, corresponding to singular points of the travel-time curve* (*in the sense determined above*).

Some of the singular points lying on the real axis t can be removed from the equation (3).

The question of removable singular points has been discussed in detail by Riesenkampf.

2. What has been said here indicates how the method of Polubarinova-Kotschina can be applied to problems where there are point sources (drains) inside the region of the flow. In this case, after plotting the travel-time curve, it is necessary to take account of all the singular points spoken here of, and add to them, moreover, the upper half-plane points corresponding to the sources

(drains) and their symmetrical points in the lower half-plane with respect to the real axis.

Riesenkampf ([5]) has shown that the latter points are removable.

Let us have by way of example some problems on ground water motion, for which the travel-time curve is one-sheeted. Place a solitary source (or drain) in the region of the flow. Corresponding to it there will appear on the travel-time curve a second sheet or; in other words; two branching points. Compared to the initial problem, equation (3) must acquire three singular points in the upper half-plane of t (two singular points corresponding to the branching points with the indices 0, 2 and one singular point corresponding to the source with the indices 0, —1) and three in the lower half-plane (with the respective indices). So we shall have six new singular points with indices making up a sum of six which is all just as it should be.

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THERMAL ENGINEERING

USE OF COMPLEX NUMBERS IN SOLVING PROBLEMS OF HEAT TRANSFER BY PLANE THERMAL WAVES

By A. M. SHKLOVER

(Communicated by A. N. Kolmogoroff, Member of the Academy, 22.III.1944)

Applied to harmonic thermal waves, Fourier's general differential equation of heat conduction

$$\frac{\partial t}{\partial \tau} = a \frac{\partial^2 t}{\partial x^2} \tag{1}$$

has the following solution:

$$t = e^{\frac{2\pi\tau i}{z}} \left(A \operatorname{ch} x \sqrt{\frac{2\pi i}{za}} + B \operatorname{sh} x \sqrt{\frac{2\pi i}{za}} \right) \tag{2}$$

Generally, the investigators in order to eliminate the imaginary quantities paroceeded to real numbers, thereby creating formidable, often even unsurmountable difficulties to further progress.

We here will make use of complex numbers without transforming them to the end.

In equation (2) it will be enough to consider the second factor only, namely

$$\theta_x = A \operatorname{ch} x \sqrt{\frac{2\pi i}{za}} + B \operatorname{sh} x \sqrt{\frac{2\pi i}{za}} \tag{3}$$

We may give it the form

$$\theta_x = A \operatorname{ch} x \frac{q\sqrt{i}}{\lambda} + B \operatorname{sh} x \frac{q\sqrt{i}}{\lambda} \tag{4}$$

where $q = \sqrt{\frac{2\pi\lambda c\gamma}{z}}$ is coefficient of heat uptake by the material. Modulus (θ_x) gives the amplitude of temperature at point x, while the argument gives the shift in phase at point x with respect to a given phase.

The meaning of the symbols used throughout this paper is: λ, thermal conductivity; γ, density; c; specific heat; α_i, inside surface coefficient of heat transfer; α_o, outside surface coefficient of heat transfer; ρ; internal thermal resistance; δ, thickness.

A variety of very difficult problems can be easily solved by means of equation (4), and this I will show here by considering some examples for which no solutions have as yet been established.

I. Suppose we have a wall consisting of many layers (among their number are layers of air). In the air adjoining the outward side of the wall harmonic oscillations of temperature are given, while on the other (inward) side the temperature of the air is supposed to be constant. The problem is to determine the damping of temperature oscillations in the wall.

The layers are counted in the direction of the x axis which is from inside outward, reverse to the direction of the motion of the wave.

a) We begin the computations with the first layer.

Equation (4) for $x=0$ gives

$$\theta_1^{start} = A_1 \tag{5}$$

For the amplitude of the heat flux at point $x=0$ we have

$$Q_1^{start} = \pm \lambda_1 \left[\frac{\partial \theta_x}{\partial x}\right]_{x=0} = B_1 q_1 \sqrt{i} \tag{6}$$

Equation (4) takes then the form

$$\frac{\theta_x}{\theta_1^{start}} = \operatorname{ch} \frac{\sqrt{x}\, q_1 \sqrt{i}}{\lambda_1} + \frac{Q_1^{start}}{\theta_1^{start} q_1 \sqrt{i}} \operatorname{sh} \frac{\sqrt{x}\, q_1 \sqrt{i}}{\lambda_1} =$$

$$= \operatorname{ch} \frac{\sqrt{x}\, q_1 \sqrt{i}}{\lambda_1} + M_1^{start} \operatorname{sh} \frac{\sqrt{x}\, q_1 \sqrt{i}}{\lambda_1} \tag{7}$$

As

$$Q_1^{start} = \alpha_i \theta_1^{start} \tag{8}$$

therefore

$$M_1^{start} = \frac{Q_1^{start}}{\theta_1^{start} q_1 \sqrt{i}} = \frac{\alpha_i}{q_1 \sqrt{i}} \tag{9}$$

For the whole of the first layer whose thickness is δ_1 the damping of the amplitude (modulus) and the shift of phase (argument) are determined by the complex expression

$$\beta_1 = \frac{\theta_1^{end}}{\theta_1^{start}} = \operatorname{ch} \rho_1 q_1 \sqrt{i} + \frac{\alpha_i}{q_1 \sqrt{i}} \operatorname{sh} \rho_1 q_1 \sqrt{i} \tag{10}$$

where $\rho_1 = \delta_1/\lambda_1$ is thermal resistance of the first layer.

All values determining β_1 are known.

b) Let us proceed to the second layer; for which we choose a separate system of coordinates with point $x=0$ at the initial surface of the layer, that is, at the surface separating the layers 2 and 1.

Making use of equation (4) for determining θ_1^{end} and Q_1^{end}; we find M_1^{end}

$$M_1^{end} = \frac{Q_1^{end}}{\theta_1^{end} q_1 \sqrt{i}} = \frac{\operatorname{sh} \rho_1 q_1 \sqrt{i} + \frac{\alpha_i}{q_1 \sqrt{i}} \operatorname{ch} \rho_1 q_1 \sqrt{i}}{\operatorname{ch} \rho_1 q_1 \sqrt{i} + \frac{\alpha_i}{q_1 \sqrt{i}} \operatorname{sh} \rho_1 q_1 \sqrt{i}} \tag{11}$$

Because $\theta_2^{start} = \theta_1^{end}$ and $Q_2^{start} = Q_1^{end}$; therefore

$$M_2^{start} = \frac{q_1}{q_2} M_1^{end} \tag{12}$$

The damping of amplitudes and the displacement of phases in the secoen layer are determined as for the first layer

$$\beta_2 = \frac{\theta_2^{end}}{\theta_2^{start}} = \operatorname{ch} \rho_2 q_2 \sqrt{} + M_1^{start} \operatorname{sh} \rho_2 q_2 \sqrt{i} \tag{13}$$

All values determining β_2 are known.

Proceeding in this way; we find the values of β for all the successive layers of the wall.

c) Let us, further, find the damping and phase shift on the transition of the wave from the outside air to the external surface of the wall (β_{air}). For the last layer of the wall denoted by the subscript n we have

$$M_n^{end} = \frac{Q_n^{end}}{\theta_n^{end} q_n \sqrt{i}} = \frac{\alpha_o (\theta_{air} - \theta_n^{end})}{\theta_n^{end} q_n \sqrt{i}} = \frac{\alpha_o}{q_n \sqrt{i}} \left(\frac{\theta_{air}}{\theta_n^{end}} - 1 \right) \tag{14}$$

Hence

$$\beta_{air} = \frac{\theta_{air}}{\theta_n^{end}} = 1 + \frac{q_n \sqrt{i}}{\alpha_o} M_n^{end} \tag{15}$$

d) A layer of the wall denoted by subscript k will be supposed to be an air filled space (air layer).

We are to find the damping and phase shift due to the transition of the wave through this layer.

To begin with, we have three equations:

$$\theta_{k-1}^{\text{end}} q_{k-1} M_{k-1}^{\text{end}} \sqrt{i} = \alpha (\theta_{\text{a. l.}} - \theta_{k-1}^{\text{end}}) \tag{16}$$

$$\theta_{k-1}^{\text{end}} q_{k-1} M_{k-1}^{\text{end}} \sqrt{i} = \alpha (\theta_{k+1}^{\text{start}} - \theta_{\text{a. l.}}) \tag{17}$$

$$\theta_{k-1}^{\text{end}} q_{k-1} M_{k-1}^{\text{end}} \sqrt{i} = \theta_{k+1}^{\text{start}} q_{k+1} M_{k+1}^{\text{start}} \sqrt{i} \tag{18}$$

From the two first equations after the elimination of $\theta_{\text{a. l.}}$ the damping and phase shift due to the air layer are obtained in the form

$$\beta_{\text{a. l.}} = \frac{\theta_{k+1}^{\text{start}}}{\theta_{k-1}^{\text{end}}} = 1 + \frac{q_{k-1}\sqrt{i}}{\Lambda_{\text{a. l.}}} M_{k-1}^{\text{end}} = 1 + M_{\text{a. l.}}^{\text{start}} \tag{19}$$

where $\Lambda_{\text{a.l.}} = \frac{\alpha}{2}$ is thermal conductance of the layer.

The value of M_{k+1}^{start}, necessary for the calculation of the layer $k+1$, is derived from the equations (18) and (19) as

$$M_{k+1}^{\text{start}} = \frac{\Lambda_{\text{a.l.}}}{q_{k+1}\sqrt{i}} M_{\text{a.l.}}^{\text{end}} \tag{20}$$

where

$$M_{\text{a.l.}}^{\text{end}} = \frac{M_{\text{a.l.}}^{\text{start}}}{1 + M_{\text{a.l.}}^{\text{start}}} \tag{21}$$

The values of $M_{\text{a.l.}}^{\text{start}}$ and $M_{\text{a.l.}}^{\text{end}}$ are introduced here solely for the sake of analogy with material layers, but they do not have the above-mentioned physical sense.

e) Obviously, the damping of the amplitude and the shift in phase due to the entire thickness of the wall are described by the modulus and argument of the following complex expression

$$\beta_s = \beta_1 \beta_2 \ldots \beta_n \beta_{\text{air}} \tag{22}$$

f) As y grows, the values of ch $y\sqrt{i}$ and sh $y\sqrt{i}$ asymptotically approach each other, and for layers with sufficiently large ρq (e. g. $\rho q \geqslant 2.0$) the formulae become simpler. Restricting ourselves to the case of a wall made up of such layers (they may be conventionally described as «thick»), we obtain the following formulae for damping and phase shift:

$$\beta_s = 0.5^n e^{\sqrt{i}(\rho_1 q_1 + \rho_2 q_2 + \ldots + \rho_n q_n)} \left(1 + \frac{q_1}{q_2}\right)\left(1 + \frac{q_2}{q_3}\right) \ldots \left(1 + \frac{q_{n-1}}{q_n}\right) \times$$

$$\times \left(1 + \frac{\alpha_i}{q_1\sqrt{i}}\right)\left(1 + \frac{q_n\sqrt{i}}{\alpha_o}\right) \tag{23}$$

Hence

$$\nu_s = \text{mod}\,(\beta_s) = 0.5^n e^{\frac{1}{\sqrt{2}}(\rho_1 q_1 + \rho_2 q_2 + \ldots + \rho_n q_n)} \left(1 + \frac{q_1}{q_2}\right)\left(1 + \frac{q_2}{q_3}\right) \ldots \left(1 + \frac{q_{n-1}}{q_n}\right) \times$$

$$\times \sqrt{\left(1 + \frac{\alpha_i\sqrt{2}}{q_1} + \frac{\alpha_i^2}{q_1^2}\right)\left(1 + \frac{q_n\sqrt{2}}{\alpha_o} + \frac{q_n^2}{\alpha_o^2}\right)} \tag{24}$$

$$\varepsilon_s = \arg\,(\beta_s) = \frac{1}{\sqrt{2}}(\rho_1 q_1 + \rho_2 q_2 + \ldots + \rho_n q_n) - \text{arctg}\,\frac{1}{1 + \frac{q_1\sqrt{2}}{\alpha_i}} +$$

$$+ \text{arctg}\,\frac{1}{1 + \frac{\alpha_o\sqrt{2}}{q_n}} \text{ (in radians)} \tag{25}$$

where ν_s is the damping of amplitudes, and ε_s is the shift of phases.

II. Let the harmonic oscillations of the air temperature with equal periods be given to either side of the wall.

In this case the amplitude and the shift of phase at each point can be obtained by summing up the complex numbers expressing for the respective point the action of the two thermal waves, each of which is so transmitted as if the temperature of the air to the opposite side of the wall was constant (which is necessary in order to satisfy the given boundary conditions).

Taking the simplest case where the wall is made of homogeneous material and at its surfaces the temperature oscillations are given and supposed, moreover, to be equal both in amplitude θ_0 and phase, we can obtain the solution directly from formula (4) if we only place the origin of coordinates in the central plane of the wall, that is in its plane of symmetry.

We have for $x = +\delta$ and $x = -\delta$

$$\theta_0 = A \operatorname{ch} \frac{\delta q \sqrt{i}}{\lambda} + B \operatorname{sh} \frac{\delta q \sqrt{i}}{\lambda}$$

$$\theta_0 = A \operatorname{ch} \frac{\delta q \sqrt{i}}{\lambda} - B \operatorname{sh} \frac{\delta q \sqrt{i}}{\lambda}$$

Hence

$$A = \frac{\theta_0}{\operatorname{ch} \frac{\delta q \sqrt{i}}{\lambda}} \tag{26}$$

$$B = 0 \tag{27}$$

Equation (4) takes the form

$$\theta_x = \theta_0 \frac{\operatorname{ch} \frac{qx \sqrt{i}}{\lambda}}{\operatorname{ch} \frac{q\delta \sqrt{i}}{\lambda}} \tag{28}$$

For this simplest case which is the only one discussed in his book Greber (1) has devoted several pages to arrive in the end at a result of rather complicated form which is, however, identical with our brief expression containing both the amplitude of the temperature oscillations and the shift in phase at point x in the form of a complex number.

III. Within an enclosure bounded by arbitrary walls of different kind there is a source responsible for harmonically varying heat convection with amplitude W_u.

Find the oscillations of temperature in the air within the enclosure. The specific heat of the inside air is neglected.

At any instant there should exist a balance between the heat evolved by the source and that absorbed by the walls.

If θ_{air} denotes the sought-for amplitude of air temperature, the amplitude and phase shift of heat absorption by every one of the walls (of surface F) will by virtue of (14) and (15) be

$$W = Q_n^{end} F = \frac{\theta_{air} q_n M_n^{end} F \sqrt{i}}{\beta_{air}} = \frac{\theta_{air} F}{\frac{1}{q_n M_n^{end} \sqrt{i}} + \frac{1}{\alpha_i}} \tag{29}$$

and the total absorption of heat by the walls

$$W_u = \sum W = \theta_{air} \sum \frac{F}{\frac{1}{q_n M_n^{end}} + \frac{1}{\alpha_i}} = \theta_{air} P \tag{30}$$

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$$\theta_{air} = \frac{W_u}{P} \tag{31}$$

θ_{air} is a complex number and its argument indicates the lag in phase of the air temperature as against the evolution of heat by the source.

IV. The harmonic oscillations of temperature (with amplitude θ_o) are given in the air outside an enclosure bounded by arbitrary walls of different kind. Find the oscillations of air temperature within the enclosure.

In this case the inner surfaces of the external walls are the sources of harmonic evolutions of heat inside the enclosure, and the problem is reduced to the foregoing one, though here we have to put

$$W_u = \alpha_i \theta_o \sum \frac{F}{\beta_s} \tag{32}$$

to obtain

$$\theta_{air} = \alpha_i \theta_o \frac{\sum \frac{F}{\beta_s}}{P} \tag{33}$$

The value of P is here the same as in the preceding problem, that is in calculating its value the absorption of heat by the surfaces of the walls transmitting the external oscillations should also be taken into account; this is required by the boundary conditions, since in determining the β_s (see above) the temperature of the internal air was supposed to be constant.

The argument of the complex number θ_{air} in this case shows the phase difference (lag) between the temperature of the air inside and outside the enclosure.

In this way any other similar problem can be solved as simply. In finding ch $y\sqrt{i}$ and sh $y\sqrt{i}$ we have to refer to the tables of hyperbolic-trigonometric functions, and plotting of diagrams is also advisable for simplifying the calculations.

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GEOPHYSICS

ON THE BOUNDARY LAYER OF THE ATMOSPHERE

By M. SHVETZ

(Communicated by N. E. Kochin, Member of the Academy, 20.III.1944)

1. There is a great many theoretical and experimental investigations on the question of velocity distribution in a planetary boundary layer, but no final solution has been reached as yet. Most of the authors who have been attacking this problem on theoretical lines have adopted special distribution laws (mostly, not well founded physically) for the coefficient of turbulent viscosity in the boundary layer, whereas the problem of the theory is to determine the distribution of this coefficient simultaneously with the velocity distribution.

The method for an approximate integration of the equations of motion as worked out in the present paper bears a certain analogy to Pohlhausen's method in aerodynamics. For applying it we need not know the distribution of turbulent viscosity in the layer, but can determine it approximately. And as the results obtained agree well with the empirical data; the new method surely deserves attention.

2. Let us write the equation of motion for a turbulent atmosphere in the form:

$$\frac{\partial}{\partial z}(\bar{\tau}_x + i\bar{\tau}_y) - l\rho i(\bar{u} + i\bar{v}) - i\left(\frac{\partial p}{\partial y} - i\frac{\partial p}{\partial x}\right) = 0 \quad (l = 2\omega \sin\varphi) \tag{1}$$

Here φ is the latitude of the place; ω; angular velocity of the Earth's rotation; ρ, density of the air; p; pressure; $\bar{u}$ and $\bar{v}$; component velocities of the air motion along the axes ox and oy, respectively; $\bar{\tau}_x$ and $\bar{\tau}_y$; component of stresses of turbulent friction along the same axes. The origin of coordinates is placed at the surface of the Earth and the axis oz is defined to be directed vertically upwards.

Define the ox axis to be so directed that $\partial p/\partial x = 0$. Adopting the notations

$$-\frac{1}{l\rho}\frac{\partial p}{\partial y} = U; \quad \bar{\tau}_x + i\bar{\tau}_y = \bar{\tau}; \quad \bar{u} + i\bar{v} - U = \bar{w} = \bar{w}_x + i\bar{w}_y$$

we shall have

$$\frac{\partial \bar{\tau}}{\partial z} = il\rho\bar{w} \tag{2}$$

instead of equation (1).

The viscosity stress $\bar{\tau}$ is linked up with the velocity by the relation

$$\bar{\tau} = \mu\frac{\partial \bar{w}}{\partial z} \tag{3}$$

where $\bar{\mu}$ is the dynamic coefficient of turbulent viscosity. Denote by δ the thickness of the turbulent layer to be determined. At the upper boundary of the turbulent layer ($z=\delta$) we have $\bar{u}=U$, $\bar{v}=0$. Hence $\bar{w}=0$; $\partial\bar{\tau}/\partial\bar{z}=0$; besides; $\bar{\tau}(\delta)=0$. At the surface of the Earth $\bar{u}=\bar{v}=0$; hence $\bar{w}=-U$ or $\partial\bar{\tau}/\partial\bar{z}=-l\rho iU$.

Let us introduce dimensionless quantities by means of the relations:

$$\bar{z}=\delta z; \quad \bar{\tau}=\rho U^2\tau; \quad \bar{w}=Uw; \quad \bar{\mu}=\frac{\rho U^2}{l}\mu$$

Equations (2) and (3) take the form

$$\frac{\partial\tau}{\partial z}=i\theta w \tag{4}$$

$$\tau=\frac{\mu}{\theta}\frac{\partial w}{\partial z} \quad \left(\theta=\frac{l\delta}{U}\right) \tag{5}$$

Differentiating equation (4) with respect to z and making use of (5), we obtain

$$\mu\frac{\partial^2\tau}{\partial z^2}=i\theta^2\tau$$

This equation gives

$$\frac{\mu}{\theta^2}=-\frac{\tau_y}{\partial^2\tau_x/\partial z^2} \text{ and } \frac{\mu}{\theta^2}=\frac{\tau_x}{\partial^2\tau_y/\partial z^2}$$

Consequently, there should have place the relation

$$\mathrm{Re}(\tau\bar{\tau})=0 \tag{6}$$

where $\bar{\tau}$ is a function conjugate with τ.

Integrate (6) over z from 0 to 1; then, taking the boundary conditions into account, we find

$$\theta\tau_y(0)-\int_0^1|\tau'|^2\,dz=0 \tag{7}$$

By virtue of (4) the latter expression can be rewritten thus:

$$\tau_y(0)-\theta\int_0^1|w|^2\,dz=0$$

Integration of (4) over z from 0 to 1 gives

$$\tau_y(0)=\theta\int_0^1 w_x\,dz$$

From the two last equations we obtain the integral relation

$$\int_0^1(w_x^2+w_y^2-w_x)\,dz=0$$

3. Let us represent the unknown profile τ by a curve satisfying some chosen boundary conditions. For the velocity near the Earth we adopt a logarithmic distribution. Then the viscosity stress can be expressed approximately thus:

$$\tau=\tau_0+a_0(z+z_0)\ln\frac{z+z_0}{z_0}+\sum_{j=1}^{n}a_jz^j \tag{8}$$

τ_0 and a_j are to be determined from the boundary conditions which have the form:

1) for $z=0$

$$\frac{\partial \tau}{\partial z} = -i\theta, \qquad \mu(0)\frac{\partial^2 \tau}{\partial z^2} = i\theta^2 \tau \tag{9}$$

2) for $z=1$

$$\tau = \frac{\partial \tau}{\partial z} = 0 \tag{10}$$

Let one more boundary condition be imposed in addition to (10):

3) for $z=1$

$$\frac{\partial^3 \tau}{\partial z^3} = 0 \tag{11}$$

The roughness of the underlying surface will have its influence allowed for by the introduction into (8) of a dimensionless «roughness»

$$z_0 = \frac{h}{s\delta} = \frac{lh}{sU\theta} = \frac{b}{\theta} \qquad \left(b = \frac{lh}{sU}\right)$$

acting in such a way that the turbulent viscosity at the surface of the Earth is different from zero. Here h is the height of the roughness and s is a number which, according to Prandtl, is equal to 30; for the Earth's surface this number has experimentally been found by Paeschke (1) to equal 8 approximately.

Put $n=3$ in (8). Making use of the five boundary conditions (9)—(11) and putting $1+z_0 \approx 1$, we obtain the following system of equations for determining the coefficients:

$$\left.\begin{aligned} &\tau_0 - a_0 \ln z_0 + a_1 + a_2 + a_3 = 0 \\ &-a_0 \ln z_0 + a_0 + a_1 + 2a_2 + 3a_3 = 0 \\ &-a_0 + 6a_3 = 0; \quad a_0 + a_1 = -i\theta; \quad \frac{a_0}{z_0} + 2a_2 = \frac{i\theta^2 \tau_0}{\mu(0)} \end{aligned}\right\} \tag{12}$$

Note for convenience

$$\frac{i\theta^2 \tau_0}{\mu(0)} = \frac{\lambda}{z_0} = \frac{Re^{i\psi}}{z_0} \tag{13}$$

Subtracting the second equation of system (12) from the first, we get

$$\tau_0 - a_0 - a_2 - 2a_3 = 0.$$

From this and the last three equations of system (12) we find

$$a_0 = \lambda; \quad a_1 = -\lambda - i\theta; \quad a_2 = \tau_0 - \frac{4}{3}\lambda; \quad a_3 = \frac{\lambda}{6} \tag{14}$$

Substituting the found values of the coefficients in the first equation of the system (12), we obtain:

$$2\tau_0 - \lambda\left[\ln z_0 + \frac{13}{6}\right] = i\theta \tag{15}$$

Near the Earth the force of Coriolis exerts no significant influence. The velocity of the wind; though sharply increasing in magnitude; does not change direction. Therefore the viscosity near the Earth can be determined from the well-known expression of Prandtl and Karman (2):

$$\bar{\mu} = \varkappa^2 \rho \left|\frac{d\bar{w}}{d\bar{z}}\right|^3 : \left|\frac{d^2\bar{w}}{d\bar{z}^2}\right|^2 \tag{16}$$

where $\varkappa = 0.36$ is an experimental constant. In our case

$$\mu(0) = \varkappa^2 |\lambda| z_0 \tag{17}$$

From (13) we find that by virtue of (16)

$$\tau_0 = -\frac{i\varkappa^2}{\theta^2}\lambda|\lambda| = -\frac{iR^2\varkappa^2}{\theta^2}e^{i\psi} \tag{18}$$

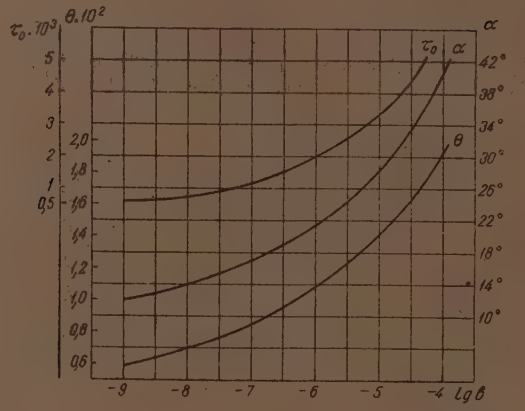

Fig. 1

Substituting the value of τ_0 in (15) and separating the real from the imaginary part, we get two relations

$$\frac{2R^2\varkappa^2}{\theta^2} = -\theta\cos\psi \tag{19}$$

$$R\left[\ln z_0 + \frac{13}{6}\right] = -\theta\sin\psi \tag{20}$$

from which R and θ and, therefore, τ_0 also, can be expressed through the angle ψ. Equation (6) is satisfied only at the boundaries. Let us put up the demand that it be satisfied in the mean, or, in other words, that the relation (7) be fulfilled. This relation by virtue of (19) can be rewritten thus:

$$\frac{\theta^2\cos^2\psi}{2} = \int_0^1 |\tau'|^2\, dz \tag{21}$$

It is this last relation that will serve for determining ψ. Substituting the value of τ' in (21), we find:

$$\theta^2\cos^2\psi = R^2\, 8.76 \tag{22}$$

and making use of (19) obtain in the end

$$\cos\psi = -\frac{8.76}{2\varkappa^2}\theta \tag{23}$$

For determining θ we obtain the equation

$$\left(\ln b - \ln\theta + \frac{13}{6}\right)^2 = \frac{2\varkappa^2}{\theta^2}\left(\frac{2\varkappa^2}{8.76} - \frac{8.76}{2\varkappa^2}\theta^2\right)$$

After θ is found, the friction $|\tau_0|$ can be determined from the formula

$$|\tau_0| = -\frac{\theta\cos\psi}{2}$$

The results of calculation within a rather wide range of variation of b are represented graphically in Fig. 1. Here α is the angle between the direction of the surface wind and the gradient wind.

4. The velocity of the wind is determined from equation (4). We have

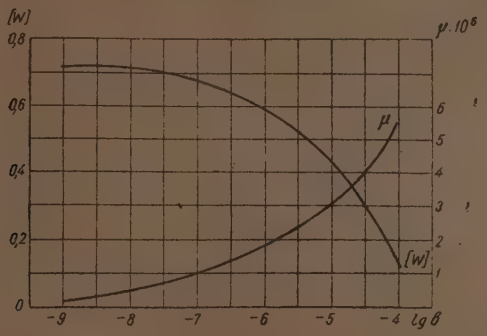

Fig. 2. Dimensionless velocity and viscosity at the height $z=0.01$

$$u = z\cos^2\psi - \frac{\ln(z+z_0) - \frac{8}{3}z + \frac{z^2}{2} - \ln z_0}{\ln z_0 + \frac{13}{6}}\sin^2\psi$$

$$v = -\frac{\sin^2\psi}{2}\left[z + \frac{\ln(z+z_0) - \frac{8}{3}z + \frac{z^2}{2} - \ln z_0}{\ln z_0 + \frac{13}{6}}\right]$$

At the surface $|u+iv| = \frac{\sqrt{\bar{\tau}_0}}{\varkappa}\ln\frac{z+z_0}{z_0}$ or, proceeding to dimensional velocity

$$|\bar{u}+i\bar{v}| = \frac{1}{\varkappa}\sqrt{\frac{\bar{\tau}_0}{\rho}}\ln\frac{\bar{z}+\bar{z}_0}{\bar{z}_0} \qquad (24)$$

According to (16) the dimensionless viscosity near the Earth is defined by

$$\bar{\mu} = \varkappa\rho\sqrt{\frac{\bar{\tau}_0}{\rho}}(\bar{z}+\bar{z}_0) \qquad (25)$$

Elimination of $\sqrt{\bar{\tau}_0/\rho}$ from (24) and (25) gives the relation between the velocity and the viscosity:

$$\bar{\mu} = \varkappa^2\rho\frac{\sqrt{\bar{u}^2+\bar{v}^2}}{\ln\frac{\bar{z}+\bar{z}_0}{\bar{z}_0}}(\bar{z}+\bar{z}_0) \qquad (26)$$

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PHYSICAL CHEMISTRY

THE ELECTROCHEMICAL BEHAVIOUR OF STAINLESS STEELS

II. THE ELECTRODE POTENTIALS OF STAINLESS STEELS AND THEIR COMPONENTS IN NITRIC ACID

By G. V. AKIMOV, Corresponding Member of the Academy, and V. P. BATRAKOV

A study of the electrochemical behaviour of stainless steels in HNO_3, besides being of theoretical interest, is of great practical importance.

This investigation is concerned with a study of the dependence of the electrode potentials of the stainless steels 16 Cr and 18 Cr—8 Ni, as well as of their components—iron; chromium, nickel—upon the acid concentration in the range 0.0001 to 10 *N*.

Furthermore, for comparison the electrode potential of platinized platinum in the acid was also investigated.

The composition of the materials studied is given in Table 1.

Table 1

Composition of Investigated Materials

Electrode	Chemical composition in %							Treatment
	C	Cr	Ni	Mn	Si	S	P	
Stainless steel 16 Cr (rods)	0.07	15.9	0.82	0.06	0.06	0.02	0.018	Forged at a temperature of 850—900° C
Stainless steel 18 Cr—8 Ni (sheet)	0.08	17.7	8.8	0.56	0.44	0.004	0.011	Tempered in plant conditions at 1000—1050° C
Iron «Armko» type (sheet)	0.05	none	none	0.10	traces	0.011	0.010	Normalized at 915—925° C
Chromium (crystals)	Electrothermal «Kahlbaum»							
Nickel (sheet)	Pure «Kahlbaum»							
Platinum (sheet)	Pure «Platinopribor», platinized							

All samples, with the exception of the platinum, were pickled in 1 *N* or 5 *N* HCl for a period of 15 minutes, thoroughly washed with warm water and then placed in a drying oven at 120—130° to dry for a period of three hours.

The difficulties connected with measuring the potential in nitric acid of high concentration were overcome by employing a calomel electrode in the following cell:

Investigated electrode /x*N* HNO_3/0.01 *N* HNO_3/
/saturated KCl/saturated KCl, $HgCl_2$,Hg

According to a calculation made with the Henderson formula, the diffusion potential at the boundary 0.01 *N* HNO_3/saturated KCl is equal to 1.5 mV; the diffusion potential at the boundary x *N* HNO_3/0.01 *N* HNO_3 may be calculated with the Nernst formula.

The HNO_3 solution with the electrode under investigation was connected with the 0.1 N HNO_3 solution by a liquid bridge with a closed ground joint. Corrections for the diffusion potentials are found to be quite considerable, amounting to 117 mV for 10 N HNO_3. All potentials measured were recalculated with respect to the hydrogen electrode. The measurements were made with the aid of a cathode voltmeter.

From 2 to 4 parallel experiments were run off.

In analysing the experimental data it should be kept in mind that an increase in the HNO_3 concentration leads, on the one hand, to an increase in the oxidizing ability of the solution (rH), and, on the other hand, to an increase in the concentration of hydrogen ions. The increase in the oxidizing ability of the solution raises the potential of the cathodic areas and consequently, the total measured potential.

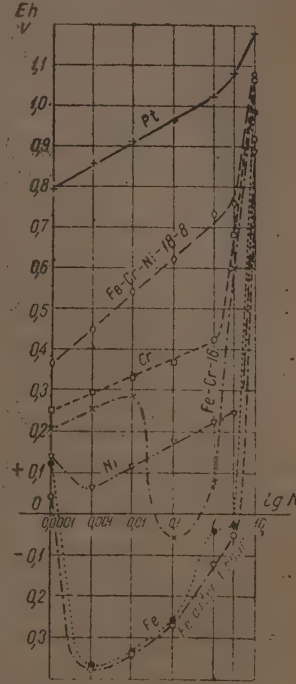
The hydrogen ions are a depassivating factor and therefore the increase in C_{H^+} may lead to a lowering in the potential.

In Table 2 and the figure are given the results of our measurements. From the tabulated data it follows that the potentials measured, with few exceptions, change little with time.

The curves given in the figure representing the dependence of the electrode potentials of the stainless steels, as well as of their components and of platinized platinum, upon the concentration of HNO_3, enable us to draw the following conclusions:

1. With an increase in the concentration of HNO_3 the potential of the iron electrode at first decreases considerably (by 0.4 V upon passing from 0.0001 N to 0.001 N HNO_3), then rises slowly and, finally, rises sharply in the region close to 10 N acid.

The potential drop is probably connected with the solution of an initial passivating film. The slow rise in the total measured potential corresponds to the region of intensive corrosion. Here the rise in potential is connected both with an increase in rH and with the polarization of the electrode. Finally, at an acid concentration of the order of 10 N transition to the passive state begins, and there is a jump to a highly positive value (nearly by 1 V upon transition from 3 N to 10 N HNO_3).



Electrode potentials of stainless steels and their components in nitric acid

2. Nickel qualitatively behaves similar to iron, but its potentials lie considerably higher. Corrosion of nickel in HNO_3 was likewise found to be weak in all concentrations of the acid. It should be noted that the potentials of nickel in 10 N HNO_3 are somewhat lower than those for all the other metals.

3. Chromium remains passive in all the acid concentrations. The course of the potential curve for chromium follows quite closely that for platinum, except for the fact that the curve lies almost entirely in a considerably more negative region.

4. Up to a concentration of 0.01 N HNO_3 the curve for steel high in chromium (16 per cent Cr) follows that for chromium, although it is approximately 0.05 V below the latter. In the interval from 0.01 N to 1 N HNO_3 the curve takes a deep drop, revealing a transition of the steel into the active state. In this interval the activating action of the H^+ ions probably prevails over the passivating action of the solution.

From the concentration 1 N HNO_3 the curve rises abruptly, pointing to the rapid transition of the steel into the passive state.

Table 2

Electrode Potentials of Platinized Platinum, Stainless Steels and their Components in Nitric Acid of Different Concentrations with Respect to the Normal Hydrogen Electrode

Electrode	Time from start of exper.	Concentration of HNO_3 (*N*)							Number of experim. averaged
		0.0001	0.001	0.01	0.1	1	3	10	
Platinized platinum[1]	5 min.	0.794	0.860	0.915	0.967	1.024	1.078	1.182	
	10 »	0.795	0.859	0.914	0.968	1.023	1.083	1.182	
	15 »	0.797	0.860	0.912	0.957	1.023	1.083	1.181	4
	30 »	0.799	0.859	0.911	0.965	1.022	1.083	1.183	
	1 hour	0.804	0.856	0.912	0.963	1.021	1.083	1.184	
Chrome-nickel steel 18—8[2]	5 min.	0.362	0.411	0.538	0.617	0.736	0.773	1.084	
	10 »	0.368	0.448	0.542	0.621	0.734	0.761	1.077	
	15 »	0.372	0.454	0.548	0.620	0.733	0.756	1.076	4
	30 »	0.377	0.465	0.550	0.634	0.726	0.746	1.047	
	1 hour	0.389	0.479	0.551	0.626	0.709	0.740	1.041	
Chromium steel (16%Cr)[3]	5 min.	0.209	0.259	0.284	—	0.079	0.521	1.077	
	10 »	0.204	0.256	0.286	—0.045	0.082	0.600	1.080	
	15 »	0.204	0.261	0.290	—0.038	0.086	0.646	1.086	3
	30 »	0.228	0.270	0.324	—0.030	0.089	0.765	1.090	
	1 hour	0.250	0.267	0.328	—0.022	0.095	0.840	1.096	
Chromium[4]	5 min.	0.272	0.292	0.320	0.362	0.414	0.680	1.060	
	10 »	0.252	0.296	0.332	0.368	0.425	0.683	1.066	
	15 »	0.261	0.302	0.331	0.369	0.428	0.687	1.067	2
	30 »	0.267	0.315	0.339	0.371	0.440	0.713	1.079	
	1 hour	0.275	0.313	0.348	0.370	0.457	0.737	1.093	
Nickel[5]	5 min.	0.150	0.081	0.115	0.184	0.220	0.244	0.893	
	10 »	0.138	0.063	0.115	0.178	0.221	0.246	0.888	
	15 »	0.136	0.053	0.119	0.175	0.220	0.246	0.884	2
	30 »	0.148	0.045	0.119	0.176	0.220	0.244	0.887	
	1 hour	0.152	0.034	0.125	0.174	0.220	0.239	0.903	
Iron[6]	5 min.	0.112	—0.364	—0.330	—0.256	—0.018	—0.011	1.020	
	10 »	0.122	—0.365	—0.331	—0.258	—0.037	—0.027	0.988	
	15 »	0.118	—0.367	—0.332	—0.260	—0.064	—0.054	0.975	2
	30 »	0.072	—0.372	—0.333	—0.264	—0.100	—0.057	0.943	
	1 hour	0.042	—0.374	—0.336	—0.265	—0.120	—0.049	0.920	

[1] No changes in samples and solution.

[2] No changes in samples and solution.

[3] No apparent changes in samples and solution. In some cases the potential in the 0.01 *N* solution attained a negative value after 1 hour, equal to—0.050 V. In the 0.1 *N* solution the mean value for the first 5 min. was not calculated, in view of the fact that in the experiments the potential attained the two extreme values,—0.050 V and+0.320 V.

[4] No changes in samples and solution.

[5] At the end of the experiment the samples were covered with elongated dark bands. Upon immersion in 10 *N* HNO_3, solution of the samples was observed in the first 1—2 min., after which the passive state set in.

[6] In solutions of average concentration violent solution of the samples was observed. The colour of the 0.001 *N*, 0.01 *N* and 0.1 *N* solutions is yellow. In the 1 *N* and 3 *N* solutions a brown precipitate was observed to form, which dissolved in excess acid. In the 10 *N* solution «periodic» phenomena were observed at the start, after which the passive state set in.

5. Highly significant is the fact that the potential curve for the steel 18—8 lies not only above that for steel high in chromium, but also above that for pure chromium. Steel 18—8 is found to be passive in all the HNO_3 concentrations studied.

The potential of steel 18—8 depends markedly upon the manner in which the sample is prepared.

We regard the electrode potentials of metals in nitric acid as complex potentials of the film—pore type. On the film (cathodic areas) a potential is set up corresponding to the oxidation-reduction potential of the solution. In the pores (anodic areas) the potential is close to the value of the potential of the active metal. The value of the total (measured) potential depends upon the values of the potential of the film and the pore, the ratio between the film and pore areas and the shape of the polarization curves. There is no doubt that the different adsorption ability of the surface of the metal with respect to the oxidizing agent in the solution may likewise influence the magnitude of the electrode potential. However, at the present time we are not as yet in a position to appraise the rôle of this factor.

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GEOCHEMISTRY

TIN IN THE COALS OF THE KUZNETSK BASIN

By S. A. BOROVICK and V. M. RATYNSKY

(Communicated by W. I. Vernadsky, Member of the Academy, 6.III.1944)

While studying the ash part of the coals of the Soviet Union and examining the spectrographs of the ash of coals from a number of deposits of the Kuznetsk basin, Urals, Central Asia, Caucasus *etc.*; the authors became aware of the fact that certain coals of the Kuznetsk basin were rather high in tin.

The presence of tin in the ash of coals has been pointed out by V. M. Goldschmidt (1). The highest concentration of tin; amounting to 0.04 per cent to ash, has been recorded by V. M. Goldschmidt in the course of a spectroscopic study of some specimens of coals from Western Germany and England; whose ash content did not exceed 2 per cent.

Until recently there were no data as to the concentration of tin in the coals of the Soviet Union. The present authors have found tin in the coals of the Kuznetsk basin in quantities approaching the maximum known for coals.

To analysis were subjected average samples from a number of coal seams of the Kuznetsk basin, often possessing a high ash content (see table); owing to this, the concentration of tin recorded becomes of still greater interest.

The coals subjected to analysis were converted into ashes in a muffle furnace at a temperature not above 600° C and studied by the method of spectrum analysis.

As a source for exciting the spectrum use was made of the flame of a voltaic arc between preliminarily purified carbon electrodes. The voltaic arc was supplied from a 220 volt current and ignited with an activator. The spectrographs were taken with the aid of a Zeiss quartz spectrograph.

The content of the individual elements was evaluated from the intensity of the spectrum lines.

The results are summarized in the table.

Deposit or mine	Shaft	Seam	Ash content, %	Tin content, %
Leninsky mine	Shaft A	Maximovky	6.8	~0.01
» »	» »	Polenovsky	10,7	~0.001
» »	Kirov shaft	Breyevsky	22.7	~0.03
» »	» »	Snyatkov	13.6	~0.03
» »	» »	Tolmachevsky	6.8	~0.03
» »	» »	Yemelyanovsky	18.1	<0.001
» »	» »	Boldyrevsky	14.0	~0.003
» »	7th November shaft	Merenkovsky	11.8	~0.003
Chertynsky deposit	1st of May shaft	Fifth	12.8	~0.03
Belovsky »	Pioneer shaft	»	4.2	~0.03

An examination of the spectrographs for the presence of tin showed that the average samples from a number of seams contained this element at an amount of about 0.03 per cent to ash.

The tin content is not in agreement with the ash content of the coal seams. Thus, for instance, the Snyatkov seam from the Leninsky mine shows an ash content of 13.6 per cent and a tin quantity of about 0.03 per cent to ash, while the Maximovsky seam from the same mine, with an ash content of 6.8 per cent, contains about three times less tin.

The concentration of tin in the coals is confined to certain seams and persists over a definite distance. Thus; the ash of seam Fifth from the 1st of May shaft of the Chertynsky deposit, the ash content of which is at 12.8 per cent, is about as rich in tin as that from seam Fifth from the Pioneer shaft of the Belovsky deposit, the ash content of which is 4.2 per cent.

It is noteworthy that all the coal seams with a high tin content are confined to the region of the Salair range.

Until now no tin deposits of commercial value have been discovered in the Salair region.

The very high concentration of tin, approaching a commercial one, in the ash of coals from a number of seams of the Kuznetsk basin furnishes new data as to the occurrence of tin in this region and may be used for a new method in search work.

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GEOLOGY

ZONAL DISTRIBUTION OF IRON, NICKEL, COBALT AND MANGANESE MINERALS IN THE WEATHERING CRUST OF SERPENTINES IN THE URALS

By B. P. KROTOV

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It has been shown by A. E. Fersman (6), B. P. Krotov (1936), M. N. Godlevsky (2) that the pH and rH of the medium in the weathering crust changes with depth, and that these changes obey a definite law; the pH gradually increasing and the rH decreasing in downward direction. Such physical-chemical conditions of the medium result in a zonal distribution through the weathering crust of the mineral matters forming from solutions infiltrated in the weathering crust and accumulated in its deeper sections.

A regular distribution in the weathering crust of different mineral matters thus formed is very distinctly manifested when viewing the place of occurrence of iron; nickel, cobalt and manganese minerals in contact-karst deposits and in the nontronite weathering crust of serpentines in the Urals. The investigations of nickel deposits carried out in the Urals in the recent years do not only confirm the above regularity; but permit to trace their subsequent alterations caused by a later lowering of the ground-water level and the sinking of separate zones with different physical-chemical conditions of the medium.

The nickel ore deposits in karst and tectonic depressions at the contact of limestones and serpentines in the Ufaley region; Middle Urals; have been studied within the recent years by a number of workers (3–5). It has been established that in the mass of the deluvium filling such depressions, iron; nickel, cobalt and manganese minerals have been precipitated at the present and some earlier epochs from the solutions infiltrated and distributed at a different depth, according to the pH and rH of the medium.

When examining the places of occurrence of the diverse compounds of these metals throughout the mass of the deluvium; it becomes obvious that their position is in full agreement with the sequence of deposition of hydroxides and hydrosilicates according to the pH and rH values of the medium; as determined by A. E. Fersman (6) and Britton (1932). The vertical change of the pH and rH of the medium in the mass of the deluvium and the concentrical zonal change of the pH of the medium around the blocks and accumulations of blocks of serpentine and limestone embedded in the mass of the deluvium; is responsible for the fact that the metals listed above are deposited individually and have different zones of distribution in the deposits of the Ufaley region. However, a subsequent lowering of the ground-water level and a rise in the oxygen potential have brought about in the upper parts of the deposits a partial change of mineral composition, connected with a change of valency and recrystallization of the initially precipitated substance from the periphery of individual concretions, or else their complete change.

In the upper part of the deposits, in a medium rich in oxygen from the solutions of $Fe(HCO_3)_2$, $Mn(HCO_3)_2$, $Ni(HCO_3)_2$ and $Co(HCO_3)_2$; at a relatively

low pH value, $Fe(OH)_3$ is precipitated. At a somewhat lower level and a higher pH value (about 5), $Mn^{\cdots}$ and $Co^{\cdots}$ are precipitated. There are first formed colloidal masses of asbolane-wad, later on, upon a lowering of the ground-water level and an increase in the pH, the latter recrystallize first to manganite and then to pyrolusite. Cobalt is admixed to manganese in asbolane-wads, the formula of which is usually given as Co_3O_4(6). Pyrolusite, in the case of a hampered drainage of waters, is frequently formed on concretions of iron hydroxides. In some cases a higher nickel content is associated with the presence of manganese, possibly in the form of $Ni^{\cdots}$.

In deeper parts of the deposits beginning with a medium with pH = 5.3 to pH = 6.8, $Ni(OH)_2$, $Co(OH)_2$ and $Fe(OH)_2$ are precipitated and concentrated in a zone poorer in oxygen. Still lower down and nearer the limestones and serpentines, where the pH value reaches 7, and the rH value becomes still lower, is precipitated the bulk of the nickel ores in the form of nickel hydrosilicates.

Finally, near the very limestones, where the pH reaches a value of 8.4, and in the mass of weathering serpentines, where the pH attains 10.5 and oxygen is almost entirely lacking, there takes place precipitation of $Mn(OH)_2$ (the pH of the precipitation is 8.5—8.8), possibly with $Co^{\cdots}$ and $Ni^{\cdots}$. Upon lowering of the ground-water level this zone becomes, however, enriched in oxygen, the manganous hydroxides are converted to manganic hydroxides, and $Mn^{\cdots}$ passes to $Mn^{\cdots\cdot}$ ($Mn(OH)_4$), which is dehydrated and forms pyrolusite. An idea of the conditions and mode of their original precipitation may as yet be formed only from the place of their concentration, where the pH must have reached the highest value.

Hence it may be seen that the distribution of iron, manganese, cobalt and nickel minerals is in full agreement with the data available on the pH and rH of the precipitation of these substances and with our evaluation on the pH and rH at different depths during the formation of deposits of the Ufaley group. Although the rH has already been changed, as a result of the lowering of the ground-water level, and the valence of some of the elements has increased, which has brought about a change of the composition of the minerals, their place in the section points distinctly to the pH and rH they had at the time of their precipitation.

The nickel deposits of the weathering crust of serpentines represent areas of the nontronite weathering crust of these rocks. At definite points of these areas an accumulation of nickel, cobalt, manganese, iron, magnesium and calcium has taken place. The weathering crust itself is known to have been formed as a result of the work of percolating and ground waters upon the weathering serpentine which in the changed parts assumes the following structure (in ascending order): (1) serpentine, (2) leached serpentine, (3) nontronitized serpentine and nontronite, and (4) ochre or clay ochre. In the first zone—of unaltered or slightly altered serpentine—there is often observed the formation of carbonates (magnesite, dolomite, aragonite). In the second zone, kerolite is formed. In the upper parts of the zone of leached serpentine and in the lower horizons of the nontronite zone nickel is concentrated in the form of hydrosilicates and hydroxides; brucite, in the shape of an accumulation of squames; quartz, in the form of veinlets. The zone of nontronite is composed of a mixture of two iron hydrosilicates—ferrimontmorillonite and ferribeidellite. The zone of ochre is mainly composed of iron hydroxides with a small amount of quartz, chlorite, *etc*. In the lower parts of the ochres nickel, cobalt and manganese are concentrated, the quantity of which gradually decreases upwards.

In the composition of individual zones take part such minerals as had been formed at different pH and rH, the minerals in the zone of leached serpentine being formed at a pH about 9—10 and without oxygen, but in the presence of CO_2; the minerals composing the zone of nontronitized serpentine and nontronite, at a pH about 7—8 and with a very small quantity of oxygen, namely: nontronite at a pH about 8, and nickel hydrosilicates, at a pH about 6.9. Along with nickel hydrosilicates there is precipitated also a certain quantity of nickel

hydroxide in the form of $Ni(OH)_2$. Nickel partly replaces magnesium in nontronites, kerolites and chlorites (1).

At the base of the zone of ochres nickel and cobalt attain a commercial content: the mode of their segregation has not as yet been sufficiently studied, but from their position within the mass of the sediments they should probably occur in the form of hydroxides (7). The presence of iron hydroxides (the pH of the precipitation from 2.3 and above) indicates the presence of a considerable quantity of oxygen (pH = 6 to 7).

The nontronitic profile of the weathering crust of serpentines is usually considered as a residual crust. However; this conception is contradicted by the process of the replacement of some areas of the rock and of some spots in antigorite needles by nontronite minerals, observed by the author in the leached serpentine. It is also in disagreement with the insignificant decrease in the volume of the rocks, incompatible with the idea of nontronite formation as a remnant of weathering. Likewise, quite inconsistent with these conceptions is the formation of nickel hydrosilicates and possibly in part also of nickel hydroxides and brucite which can by no means be remnants of weathering, but are products of precipitation from solutions. All this points to the formation of the nontronite zone in the lower part of the crust; or, in other words, in the cementation zone, with a pH from 8 upward and with a small amount of oxygen as a product formed anew from iron and nickel; and probably in part also silica, brought in the solution. The upper zone of the profile—the zone of ochre—is undoubtedly the product of a later transformation of the nontronite zone, which took place at a time when a removal of silica occurred and iron hydroxides accumulated. These processes are possible in an acid medium rich in oxygen, with a pH = 2.3—6.6. Nickel and cobalt migrated and were precipitated at the base of the zone of ochre, where the stagnant ground waters had a pH = 6—7, and where oxygen was present, most probably in the form of hydroxides.

The problems of formation of the nontronite zone require further study.

From what has been said above the following conclusions are to be drawn: (1) in the course of the present weathering under conditions of a temperate climate and of the weathering of the same serpentines at some earlier periods, it was always the same metals, Fe, Ni, Co, Mn, that were passing into solution; (2) there is a resemblance in the position of the compounds of these metals in the nontronite weathering crust and in the mass of loose deluvial products filling the depression at the contact of the limestones and serpentines; (3) the zonal sequence in the changes of these ore segregations in the vertical direction may be accounted for by the zonal distribution in the nontronite zone and in deluvial accumulations of media with different pH and rH values. The latter determined the place, the degree of oxidation and the shape of the segregation of the various hydroxides and hydrosilicates of iron, manganese, nickel and cobalt and their subsequent alteration as associated with subsequent changes of the ground-water level; (4) hydrous iron silicates (nontronites) and hydrous silicates of nickel originate and concentrate in the Middle and Southern Urals in a single zone, in the deep parts of the nontronite crust and of the deluvial accumulations, in depressions at the contact of limestones and serpentines, where the value of the pH of the media must be about 8—9.

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HYDROGEOLOGY

ON THE REGULARITIES IN THE DISTRIBUTION AND FORMATION OF UNDERGROUND WATERS

By N. K. IGNATOVICH

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1. The geological-structural and facial-lithological peculiarities of an area, along with the complex of physical-geological conditions, create the natural background for determining the trend of hydrological processes. The geological-structural and facial-lithological conditions alone are frequently inadequate to account for the distribution and formation of the hydrological elements observed at present. Continental fresh-water deposits frequently enclose highly mineralized waters; and reversely, lagoonal saline complexes sometimes yield, following definite zones and directions, fresh or slightly mineralized waters.

2. In order to understand the conditions and the nature of distribution of underground waters of a certain region, particular attention should be paid to the elucidation of one essential factor, namely, the nature of the regime and the d y n a m i c s of the underground waters (1). A hydrological analysis made on the basis of geological, physical-geological and other data, has to: (a) reveal the typical elements of dynamics and regime of underground waters; and (b) to establish the chemical peculiarities in the distribution of the various types of waters.

3. In the distribution of underground waters and in their formation definite regularities of regional nature are universally observed which are responsible for their h y d r o g e o l o g i c a l z o n i n g. The regional changes in the regime, dynamics and chemical composition of the waters are generally dependent on the structural and hypsometric position of the water-bearing complexes during a prolonged geological time interval. In this sense it would be sufficient to compare the highly «flushed» parts of the southern and western slopes of the Moscow depression, rich in fresh waters, with the inner portions of the same area, which from the hydrogeological standpoint are nearly motionless and unflushed.

4. The best pronounced zonal distribution of underground waters is recorded in the areas of an extensive development of sedimentary formations of great thickness. Most frequently, three hydrogeological zones may be established here; each of these shows a complete agreement between the dynamical conditions of the underground waters, their underground drainage and chemical composition. As it will be seen from the table adjoined, the comparison may be extended further, to the establishment of the genetic types of the waters, their geochemical position and even their relation to the fundamental problems of economy.

5. The zonal distribution of underground waters is traceable both in the vertical and horizontal direction; in the former case the part played by the degree of the dynamics of the underground waters in their distribution and formation is especially conspicuous. In the latter case, that of shallow-lying aquiferous complexes, the leading part belongs to the surface factors.

Scheme of Zonal Distribution and of Conditions

Geochemical zones	Geohydrodynamic zones		Genetic types of waters and nature of resources of underground waters	Hydrogeochemical
	Formation of water regime and of underground drainage	Geological-hypsometric position of zones		Nature of process
Zone of hypergenesis	Zone of active water exchange (underground drainage takes part in active water exchange with surface)	Actively flushed parts of structure. Zone influenced by river network drainage (sometimes to a depth of 300 m and more)	Waters of a recent meteoritic origin, in motion. Dynamic resources prevail over static ones	Active leaching-out of chlorides and sulphates, decreasing from elevations to depressions, and in direction of growing aridity
	Zone of impeded circulation of underground waters (underground drainage difficult, water exchange sluggish)	Deep portions of flowing artesian basins, at platforms to a depth of 500—600 m; in folded areas with tectonic disturbances, to 1000—2000 m (thermal waters)	Waters slowly renewed, replacing more ancient ones. Static resources prevail over dynamic ones	Salt complex slowly washed out of the rocks (marine, lagoonal and other facies); profile of water is changed within a long period
Zone of katagenesis	Zone of stagnant underground waters (underground drainage manifest only on a geological time scale)	Usually in the zone of deep occurrence of sedimentary complex, but in hydrogeologically isolated (closed) structures; position of zone possibly near surface	Waters buried often connate, usually yielding enormous static resources	Accumulation of salts, metamorphism of waters, geochemical and physical-chemical processes (diffusion, osmosis, *etc.*)

6. The upper zone of active water exchange determines the underground drainage with large dynamic resources, and is characterized mainly by fresh hydrocarbonate, sometimes sulphate waters, and in acid regions, also by sulphate-chloride waters. This zone comprises not only ground-waters, but also those artesian waters which take part in the active water exchange and underground drainage (for example, some of the horizons of the Carboniferous of the Moscow artesian basin).

7. The middle zone of hampered circulation of underground waters produces a sluggish water exchange with the surface. The waters of this zone are usually highly mineralized, particularly in poorly flushed structures, their composition being of a type transitionary between sulphate and chloride waters. Adequate hypsometric and tectonic conditions, in particular the presence of large disturbances (mountainous folded regions), permit of a deep water exchange; as a result, within this zone very young waters of alkaline type are sometimes produced (2), which are occasionally ascending thermae. Waters of this type are known to be extensively developed in oil-bearing regions.

8. The deepest hydrogeological zone is characterized by underground waters of stagnant nature. Very important for this zone are physical-chemical processes which bring about metamorphism of the water, formation through exchange reactions of chlorine-sodium-calcium waters and brines. The lack of water exchange at considerable depths results in the fact that here, owing to gravitation, diffusion, and other physical-chemical processes, waters and brines of high concentrations are formed, sometimes reaching the limit of saturation.

zones		Examples	
Shaping of chemical type of water	Part played in economy	Moscow artesian basin	Northern Caucasus
Formation of primarily fresh hydrocarbonate waters; in arid regions and in areas of depressions also sulphate and sulphate-chloride waters	Waters mainly fresh; problems of drinking-water, industrial and technical water-supply	Ground waters of Quaternary and Mesozoic deposits, and of Carboniferous, in the sphere of active water exchange (drainage in basins of Upper Volga, Oka, Moskva and Klyazma rivers). Waters fresh, hydrocarbonatic	Waters of Mesozoic rocks in the area of their high hypsometric position. Mainly fresh and low-mineralization waters
Formation of hydrocarbonate, sulphate-chloride waters. In folded young areas also alkaline and thermal, occasionally H_2S waters	Waters mainly sulphate-chloridic, alkaline, thermal; evidence of disintegrating petroleum deposits. Sphere mainly of balneological problems, and less frequently of mineralized waters of industrial value	Waters of Upper Devonian, found under conditions of slow circulation, transitionary from hydrocarbonatic to sulphatic and sulphate-chloridic	Waters of Mesozoic and Lower Tertiary deposits in piedmont zone. Circulation of underground waters hampered but deep, owing to tectonic disturbances. Composition — transitionary from sulphate-alkaline to chloride-alkaline
Formation of highly mineralized chlorine-sodium-calcium waters and brines	Commercial brines of Br, I, Ra and other rare elements: petroleum deposits	Brines of Middle Devonian of high mineralization and of high degree of metamorphization of chlorine-sodium-calcium type	Waters of Tertiary and Cretaceous deposits at transition from a folded area to platform. Waters of chlorine-sodium-calcium type

9. Waters and brines of this type are but rarely formed on the surface; and are mostly observed there (for example, in lakes) only as a manifestation of the work of ascending underground waters. In order that waters of this type be formed, a hydrogeologically isolated medium is necessary. Such conditions are not necessarily available at considerable depths only. Owing to the peculiarities of the structure and the water regime, chlorine-sodium-calcium brines are sometimes recorded at comparatively shallow depths. This is largely due to the fact that hydrologically isolated conditions of the medium have been maintained here during a long geological period.

10. It is well-known that for its formation and preservation petroleum also requires the conditions of an isolated and closed medium. But a genetic relationship between chlorine-sodium-calcium waters and petroleum; which is sometimes presumed, is not always indispensable.

11. The chlorine-sodium-calcium waters and brines which are widely developed, for instance, within the Russian platform, form large basins of stagnant, very ancient waters, often representing buried connate waters. At the present time these ancient waters are not necessarily contained in the same rocks in which they had been buried at the period of sedimentation. A slow migration of these waters from the higher places of occurrence of rocks results in a «flushing» and «aeration» of the stagnant zones within a rather long geological period. The underground drainage generally proceeds seawards or in the direction of oceanic hollows, and in some cases of large structural depressions having tectonic disturbances and «windows» for a deep, although very slow circulation of water (the Caspian depression).

12. The above-mentioned factor of the dynamics of water sometimes disturbs the regular transition of underground waters from hydrocarbonate to sulphate ones and further to chlorine ones, characterized by a gradual increase of mineralization and metamorphization with depth. The process of migration of salts in underground waters is the most active in the direction towards the zones of water exchange, *i. e.* mostly in upward direction; but in folded regions with a deep circulation of water it takes place in other directions, too,—from slightly flushed saline rocks to those with a higher permeability. As a result, below the ancient stagnant chlorine-sodium-calcium waters, in the section of rocks, circulating young alkaline, as well as mixed alkaline-chloride waters. may be encountered within a certain interval.

13. The presence of fresher waters below the horizon of mineralized waters is often explained by the burial of fresh waters at a remote geological epoch and by fully stagnant conditions of the recent fresh water horizon.

When facing such relations between fresh and mineralized waters; it is necessary to analyse the hydrodynamic conditions of the region in order to find out whether there is some water exchange along definite directions, associated with a more active flushing of rocks. In such places where water exchange is lacking, there are probably no conditions for the preservation of fresh waters either.

14. The following considerations make us call to question the presence of fresh buried waters within the Black Sea depression and other structures.

In the section of the Palaeozoic deposits of the Russian platform which is of considerable thickness (over 1500 m) and presents a highly diversified lithological picture, waters of a most various chemical composition might have originally been buried. Among these were the waters of basins with a normal salinity, as well as freshened and oversaturated with salt. As seen from the recent section, the original chemical properties of the waters have not been preserved. As the laws of dynamics do not hold for the deeper zones, the laws of gravitation and physical-chemical processes have brought about a regular exchange of waters according to the degree of mineralization with depth, and an accumulation in the deeper zones of brines with a high concentration.

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GENETICS

HYBRIDS BETWEEN WHEAT AND *ELYMUS*

By W. E. PISSAREV and N. M. VINOGRADOVA

(Communicated by N. V. Zyzyn, Member of the Academy, 26.VII.1944)

The attempts of producing hybrids between wheat and rye, on the one hand, and various species of *Agropyrum*, on the other, undertaken by N. V. Zyzyn(²), proved a brilliant success; and one cannot make doubt any longer that the wild relatives of our small-grains can be involved in the breeding work which is being carried on with cultivated cereals.

The genus *Elymus* is no less interesting in this respect; it embraces more than 45 species distributed throughout the moderate belt of both hemispheres. In this country the genus *Elymus* is represented by over thirty species which offer a highly diversified ecological picture.

Some of the *Elymus* species are good fodders; others are used for their grain, yielding up to 700 and more big grains per spike. The coarse-grained species of *Elymus*, for example, *E. arenarius* and *E. giganteus*, are of particular interest for purposes of hybridization with small-grains.

Aside from this they are remarkable for the high content of gluten in their grain (P. N. Shibaev).

We started our experimental crosses between small-grains and representatives of the genus *Elymus* in 1941. Preliminary experiments have shown that in the case of such crosses the breeder is frequently faced with considerable difficulties: when pollinating wheat with *Elymus* pollen, fertilization succeeds on many occasions and the formation of embryo and endosperm sets in, but their development is arrested soon. In this case there are sometimes formed enlarged ovaries and still less frequently meagre seeds with underdeveloped embryos, or having no embryos at all.

One of the agents responsible in the case of remote hybridization for the abortive development of embryos already laid down we see in the biochemical diversity of the components of these crosses. This difficulty may, however, be obviated by applying the method of mutual vegetative «blending» of the forms to be crossed; this procedure was successfully applied by Michurin to horticultural plants.

But the technique of grafting cereals has not yet been sufficiently worked out, and our attempts made in this direction also proved a failure. We have decided therefore in favour of the method of mutual transplantation of the embryos of the genera or species to be crossed on to the endosperm of the other component.

We were guided in our endeavours by the work of Schander (³) on the rôle played by the phytohormones contained in the endosperm of the grain of cereals, and on the influence they produce upon the development both of the embryo and the adult plant.

As experimental genera we chose spring wheat and rye; among the forms of these plants there are such as defy any crossing and such as give a high percentage of hybrid seed.

The procedure of transplantation consisted in the following. The embryo together with the scutellum was cut from the dry caryopsis with the help of a safety-razor blade; care was taken to cut as little endosperm as possible. The embryo was then transferred on to the «endosperm wilding» whose embryo had preliminarily been cut out. The graft-embryo was fixed in place by means of a paste prepared from a flour of the same genus to which the «endosperm wilding» belonged.

Under experiment were the spring-wheat *Lutescens* 62 which presents difficulties in crossing with rye. The embryos of *Lutescens* 62 were grafted to the endosperm of a spring-rye from East Siberia. The wheat-plants grown on the rye endosperm showed no morphological distinction from the controls. It was only the seeds that proved different. The grains of the plants grown on rye endosperm differed from the control grains by their rather dull coloration and less vitreous aspect; and, what is especially interesting, their shape diverged from the type, being somewhat angular. The furrow was wider and the grains shorter, some of them were dolioform. Chemical analysis made in the usual way revealed considerable changes in the chemical composition of the grafted grain as compared to the controls.

	In per cent to absolutely dry matter					
	Total nitrogen	Protein nitrogen	Protein	Sugars	Starch	Ash
Lutescens 62, control	2.844	2.434	13.88	3.812	60.56	2.232
Lutescens 62, grafted	3.409	2.810	16.02	4.191	55.22	2.446

Prof. A. A. Shmuk has kindly made at our request a more detailed analysis of the grain. For this purpose he had recourse to the trifructosan reaction suggested by Tillmanns (3). According to Tillmanns; Kiesel and Kretovich; trifructosan is a carbohydrate specific of the grain of rye and absent in the grain of wheat. Shmuk (4) has shown that it is equally lacking in the control grain, while 5 g of the «grafted» wheat gave 0.14 g of trifructosan. Upon being hydrolysed with 2 per cent hydrochloric acid, all reactions for ketose succeeded and the hydrolysate showed the laevorotatory nature peculiar to ketose.

The gluten isolated from the flour of the control wheat had the usual light-pink colour, while the gluten obtained from the grain of the «grafted» wheat had the colour of rye dough, *i. e.* was dark-brown. These differences in the colour of the gluten seem to be due to the different contents of tyrosin and to the different degrees of activity of tyrosinase in the grain of the «grafted» and the control wheat.

As to the causes underlying such striking chemical changes in the «grafted» wheat, we feel inclined to make responsible for them largely the phytohormones contained in the endosperm of the rye. These changes which made the chemical characters of the «grafted» wheat and of the rye closely approach each other have, in agreement with our expectation; resulted in an increased ratio of successful crosses between *Lutescens* 62 and rye.

Aside from *Lutescens* 62 we took into experiment also two varieties of wheat, hybrid 170 and «Aurora», which are also known to cross but poorly with spring rye.

The results obtained were as follows. While the control crosses of normal *Lutescens* 62 gave but 4.3 per cent of developed ovaries; the grafted plants gave successful results in 25 per cent of cases. In the variety hybrid 170 we obtained 3.4 and 11.5 per cent, respectively, and in the variety «Aurora»; 2.8 and 19.2 per cent to the number of flowers pollinated with rye pollen.

Thus, by grafting the embryos of the spring wheat on to the endosperm of spring rye the biochemical characteristic has been made to approach the latter; at the same time the procedure went also to increase considerably the percentage of the hybrid grain obtained.

This method of bringing together two poorly crossable genera was applied by us since 1942 in our hybridization experiments with wheat, barley and rye, on the one hand; and various species of *Elymus*, on the other.

Following the advice of N. V. Zyzyn; we have chosen among the species of *Elymus* two coarse-grained species, standing apart for their vigorous growth habit: *Elymus arenarius* L. and *E. giganteus* Vahl.

Let us dwell upon our experiments with wheat. Under experiment were various ecotypes of soft wheat. It has soon been found, however; that not all its forms are liable to cross with *Elymus*. The best results have been obtained with the East-Asiatic forms and their hybrids; but not unless the method of embryo-grafting was applied. We were most successful in those cases where both mother and father plants had been grafted on the endosperm of the component. Thus, spring wheat 1803, grown three times upon the endosperm of *E. arenarius* 19, when pollinated with the pollen of the latter gave 1.55 per cent of successful seeds. In those cases where the pollen had been taken from an *Elymus* plant grown on the endosperm of wheat 1803, the percentage of successful fertilization was 7.5. The spring wheat «Prélude», in spite of a repeated grafting on to the endosperm of *Elymus*; yielded no seed when crossed with the latter; but when pollinated with *Elymus* grown on wheat endosperm; the percentage of set seed was as high as 3.1. The hybrid wheat VEP 2 repeatedly grafted to *Elymus* gave; when pollinated with the pollen of common *Elymus*, 0.4 per cent of seed; while in the cases where pollen from a «grafted» *Elymus* was used, the percentage was 1.7.

From these figures it may be seen that in order to obtain more or less appreciable quantities of hybrid seed one has to emasculate and pollinate scores of wheat flowers. But the formation of hybrid seed does not yet guarantee a successful development of the hybrid plants themselves, on account of the abnormal development of the hybrid embryos. Thus; from crosses made in 1942 we obtained over 200 rather meagre grains; only four of which germinated. One of the plantlets perished soon. Out of the remaining three grains two plants were due to pollination of an early wheat variety (VEP 2) with the pollen of *E. arenarius* 19. One of the plants proved to be a typical dwarf with aciculate leaves; and perished after three months. Another (No. 80) showed a normal though very low growth, and started earing by November 1943.

The maternal parent of the third plant (No. 34) was a spring wheat of the northern early type; it has also been pollinated with pollen of *E. arenarius* 19.

Throughout the winter of 1943—44 the two plants were kept at a temperature of +5—6° C. Towards the spring only one of them; *viz.* No. 34, has survived. In the spring of 1944 this plant was transferred to a hothouse where it recovered soon; part of the ear-bearing shoots which had been laid down during the winter period, formed spikes as early as by May.

When examined cytologically(2), the hybrid was found to possess a chromosome-set $2n = 49$. The mother plant was *Triticum vulgare* with $2n = 42$; the father plant, *E. arenarius* with $2n = 56$.

In this way for the first time in the history of breeding of small-grains a hybrid between wheat and *Elymus arenarius* has been produced.

The hybrid plant was rather vigorous and in June 1944 it had up to 60 stalks. The leaves of the hybrid were rough and stiff like those of *Elymus*; unlike the leaves of wheat, they were directed upwards at an acute angle; as in *Elymus*. During the summer of 1943 the colour of the leaves differed but little from that of wheat and was even somewhat yellowish. But in the spring of 1944; as they grew rougher, the leaf-blades and culms assumed the waxy bloom and pronounced light-blue tint peculiar to the *Elymus*.

The spike of the hybrid showed the structure of a mixed type; the spikelets

of the upper part of the ear were inserted singly at each notch of the rachis, as in wheat. The top-notches bore 2—3 spikelets each; according to the *Elymus* type. The spike and spikelet glumes were morphologically closer to those of *Elymus*. The flowers were sterile because of the underdevelopment of their anthers.

It should be pointed out that in the next year (1943) we obtained again by the same method a considerable quantity of hybrid seed; and in the summer of 1944 we had new hybrid plants.

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EXPERIMENTAL MORPHOLOGY

SOME DATA ON REGENERATION OF FIBRILLARY STRUCTURES IN TISSUE CULTURES

By W. SUNTZOWA

(Communicated by I. I. Schmalhausen, Member of the Academy, 10.I.1944)

In the study of processes of wound healing an important place belongs to the question: whether the fibrillary structures of the connective tissue are capable of regenerating. While investigating the histogenesis of fibrillary structures in tissue cultures ([8–9]) and in the process of wound healing ([10]), we have been able to demonstrate that fibrillary structures invariably begin to form anew from fibres of the edge of the cut.

The fact that the fibrillae start growing from the edge of the wound can well be interpreted as testifying to their regenerating capacity: the most vigorous new growth of fibres both in tissue cultures and over wound areas proceeds largely at the expense of these regenerating fibrillae.

It should, however, be pointed out that the thick and vigorous collagenous fibres, both in tissue cultures and along the edge of wound areas; do never simply continue to grow, but they always undergo rearrangements; are decomposed into separate fibres, after which they give rise to regeneration of very thin fibrillae. The ability of collagenous fibres to such a rearrangement and to a subsequent regeneration has been reported by Heringa([5]) and Gaza ([4]) for the case of wounds or cuts; and Knake ([6]) for the case of tissue cultures. Berezkina ([1]) who cut out the central portion of a successfully growing culture of the connective tissue of a chick-embryo and grafted in place of the excised bit a solid connective tissue of an adult chick; has succeeded in making grow and regenerate the ends of the collagenous fibres of the graft.

Entirely different results have been obtained by Levi, Montalcini and Sacerdote ([7]). These writers maintain that there is absolutely no genetic succession between the network of collagenous fibres of the explantate and the fibrillae formed anew in the culture: the old collagenous fibres explanted together with the fragment roll up and disintegrate; while new fibres are forming within the zone of growth quite independently of them. In his work on the development of connective tissue in transparent chambers inserted in a rabbit's ear; Stearns ([11]) utterly denied the possibility of the fibrillae forming anew by way of regeneration from the edge of a cut made through the old fibres.

In order to shed more light on this question; A. W. Rumjantzew has invited me to follow up the formation of new fibrillae in a culture with a basic central bit cut out of it.

Fischer ([3]) and Ephrussi ([2]) have shown that if a culture that develops a sluggish or abortive growth is inflicted a wound by cutting out the control portion of the basic bit or a section of it, an intense proliferation of cells will begin from the edge of the cut and continue until the whole region excised is fully regenerated.

These writers believe the process described to be induced by some substances that are liberated in the edge of cell injury and whose effect is to stimulate growth in the region of the wound. Along with the cells, fibrillary structures must undoubtedly also begin to regenerate from the edge of the cut made to excise the basic fragment. It seemed to us that these experiments can once more substantiate the results of our previous investigations, *viz.* that growth and regeneration of new fibrillae in wounds always proceeds from the edge of the cut made through the old fibres which act upon the new fibres as organizers or, more strictly speaking, as a stimulus of growth.

Material and Method. As material for our work use was made of cultures of subcutaneous connective tissue of 7—14—16-day-old chicken embryos. The cultures were grown after the method of the suspended drop; with addition of embryonic extract and plasm. Part of the cultures were grown without addition of embryonic extract; in order to make the development of these cultures lag somewhat behind the rest. Some of the cultures had the central portion of the basic fragment or a segment of the culture cut out with the aid of a sharp ocular scalpel by the second day, others by the fourth day of cultivation, which was done in order to have at our disposal ageing cultures.

After the operation, the culture was washed with Ringer solution, and a drop both of embryonic extract and of plasm was added, upon which the culture was embedded. The purpose of washing which followed the infliction of the wound was to eliminate from the wound the growth-stimulating substances which are always liberated in the case of wounding; and to slow down the process of regeneration.

On the first and second day upon the operation the cultures were fixed with 10 per cent neutral formalin impregnated with silver nitrate; according to the method of Rio-Hortega modified by Rumjantzew. Total preparations were manufactured.

The Author's Own Observations. Those of the cultures which had their central portions cut out by the second day of cultivation, showed invariably a very rapid regeneration of the central portion. As early as by the first day upon operation one could observe cellular growth to proceed towards the centre of the wound. The fibrillary ground substance of the remaining part of the explantate, which stained a deep black, was originally bounded by a sharp outline along the edge of the cut. The day next to the operation it was always the cells that began to proliferate from the edge of the cut, and it is not until the cells have organized the medium in the zone of growth adjoining the edge of the cut, that the fibrillary cultures are also involved in regeneration. The initial moment of the formation of fibres may easily be traced on our preparations. The very first isolated thin and straight fibrillae grow into the zone of growth from the fibrous ground substance of the edge of the cut. On total preparations these fibrillae are seen to be closely connected with the fibres of the stroma of the ground bit (Fig. 1).

If there is opulent growth of cells and fibrillae from the edge of the cut, the sharp outline of the edge soon becomes diffuse and the fibres of the stroma of the ground bit pass directly into very thin fibrillae which are formed anew in the zone of growth.

The first instants of the regeneration of the fibrillae are characterized by their straight sagittal growth in a direction perpendicular to the edge of the cut, as shown in Fig. 2. As the fibrillae continue growing, they change their direction, forming interlacements and assuming a different orientation according to the conditions of the medium in the zone of growth.

In none of the operated cultures did we observe the dissolution of the fibrous structures of the ground fragment, which has been described by Levi, Montalcini and Sacerdote (7): the fibres of the edge of the cut always undergo rearrangement of their ends, and these give rise to the regeneration of the fibrillary structures of the zone of growth.

After the central portion of the culture has been cut out, the fibrillary structures in the remaining part of the ground fragment frequently assume a circular arrangement along the edge of the cut. It is noteworthy that, in spite of the normal growth of the cells from the edge of the wound, we never happened to observe

Fig. 1. Culture of connective tissue of a 14-day-old chick embryo. Central portion of the culture excised by the second day of cultivation. Fixation, the day next to operation. Formalin. Impregnation with $AgNO_3$. Ocular 15, objective 40

Fig. 2. Culture of connective tissue of a 7-day-old chick embryo. Central portion of culture excised by the second day of cultivation. Fixation, the second day upon operation. Formalin. Impregnated with $AgNO_3$. Ocular 10, objective 90

in such cultures fibrous neoformations within the zone of growth, but only their increase of density along the edge of the wound. Even within a single culture it may occasionally occur that in a point of the wound where the fibres along the edge of the cut have assumed a circular arrangement; there will be no formation of fibrillae in the zone of growth, while in a point where the fibres are cut without definite order, they invariably give rise to regeneration of new fibrillae accompanying the proliferation of cells into the zone of growth. The fibrillae always start growing straightforward, at a right angle to the fibres of the ground substance of the bit.

The most illustrative are the pictures of regeneration of the fibrillary structures in ageing, two- or three-day-old cultures in which the process of growth has been suspended. In such ageing cultures the trauma inflicted used to produce on the day next to the operation an intense proliferation of cells from the edge of the wound, whereas fibrillary structures were found to regenerate at a much lower rate.

The fibrous stroma of the ground fragment transforms in ageing cultures into a network, forming loop-like structures. The fibres begin to regenerate from the edge of the wound in the form of coarse loop-like structures, constituting an indissoluble whole with the loop-like stroma of the ground fragment. The boundary between the old fibres of the wound edge and those formed anew is thereby obliterated.

Thus, our experiments give us right to maintain that the formation of new fibrillae from the edge of the wound inflicted upon the culture proceeds within

the zone of the wound after the cells have organized the medium at the expense of the regeneration of the fibrillary structures which had previously existed there.

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